Random Seas

Liu, Z.; Frigaard, Peter

Publication date:
1997

Document Version
Publisher's PDF, also known as Version of record

Link to publication from Aalborg University

Citation for published version (APA):
Liu, Z., & Frigaard, P. (1997). *Random Seas*. (1. udgave ed.) Aalborg Universitet, Inst. for Vand, Jord og Miljøteknik, Laboratoriet for Hydraulik og Havnebygning.

RANDOM SEAS

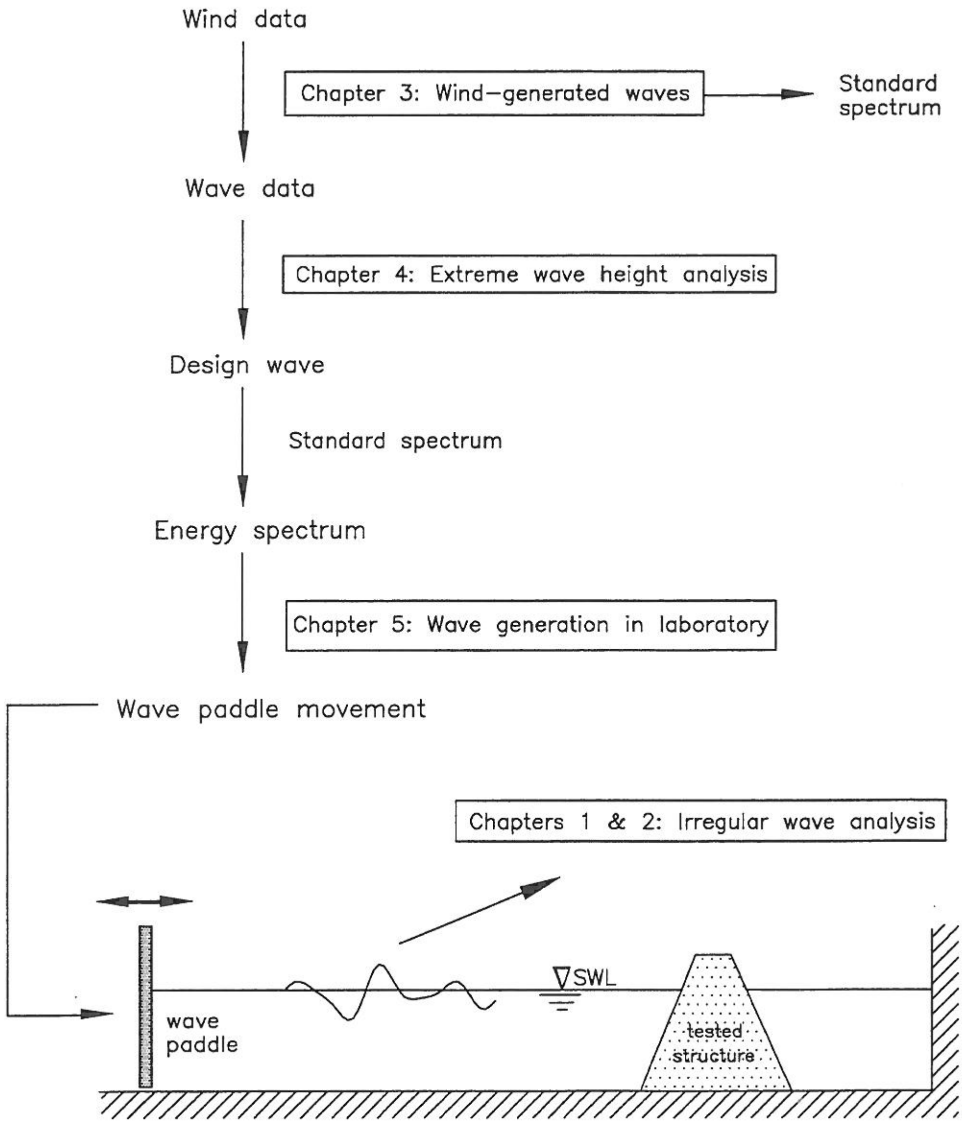
Zhou Liu and Peter Frigaard

1. udgave. November 1997
Laboratoriet for Hydraulik og Havnebygning
Instituttet for Vand, Jord og Miljøteknik
Aalborg Universitet

Preface

Sea waves are the most important phenomenon to be considered in the design of coastal and offshore structures.

Every sailor has noticed that, when wind is blowing, there are a lot of large and small waves propagating in many directions. Such waves are called *short-crested waves* because they do not have a long crest. Contrast to short-crested waves, we have *long-crested waves*, i.e. large and small waves moving in one direction. Even though there are some research efforts on short-crested waves and their effects on structures, long-crested waves dominate today's structure design. The book deals with long-crested waves. The contents of the book is illustrated in the figure.



There are two methods for irregular wave analysis, namely time-series analysis and spectral analysis, which will be dealt with in Chapter 1 and Chapter 2 respectively.

It should be stressed that, even though all contents in the book are related to sea waves, they have broader applications in practice. For example, the extreme theory has also been applied to hydrology, wind mechanics, ice mechanics etc., not to mention the fact that spectral analysis comes originally from optics and electronics.

The book intents to be a textbook for senior and graduate students who have interest in coastal and offshore structures. The only pre-requirement for the book is the knowledge of linear wave theory.

Michael Brorsen, Associate Professor at the Hydraulic and Coastal Engineering Laboratory, Aalborg University is gratefully acknowledged for the valuable comments.

Contents

1 Time series analysis I : Time-domain

The recorded time series of the surface elevation of irregular waves can be studied by either the time-domain or the frequency-domain analysis. These two analysis methods will be described in the next two chapters, respectively.

1.1 Definition of individual wave : Zero-downcrossing

Individual wave is defined by two successive zero-downcrossing points, as recommended by IAHR (1986), cf. Fig.1.

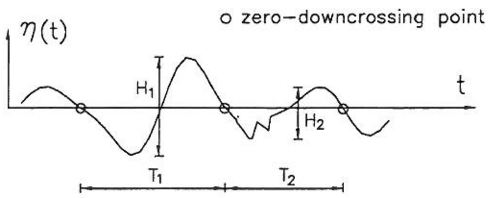


Fig.1. Individual waves defined by zero-downcrossing.

Fig.2 is an example of surface elevation recordings. The application of zero-downcrossing gives 15 individual waves (N=15). In Table 1 the data are arranged according to the descending order of wave height.

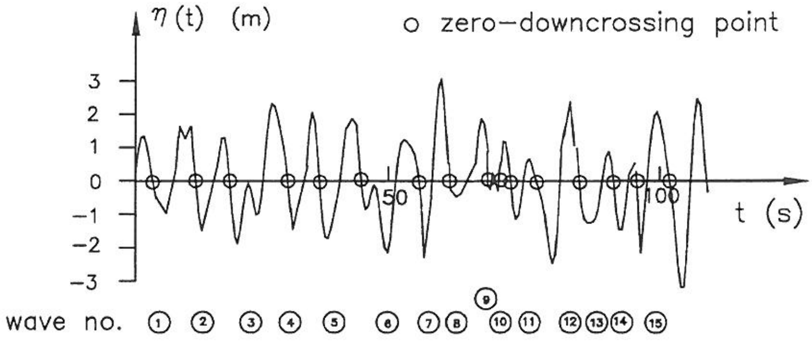


Fig.2. Application of zero-downcrossing.

Table 1. Ranked individual wave heights and corresponding periods in Fig.2.

rank i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
H (m)	5.5	4.8	4.2	3.9	3.8	3.4	2.9	2.8	2.7	2.3	2.2	1.9	1.8	1.1	0.23
T (s)	12.5	13.0	12.0	11.2	15.2	8.5	11.9	11.0	9.3	10.1	7.2	5.6	6.3	4.0	0.9
wave no. in Fig.2	7	12	15	3	5	4	2	11	6	1	10	8	13	14	9

1.2 Characteristic wave heights and periods

Usually the surface elevation recording exemplified in Fig.2 contains more than 100 individual waves. Which wave should be chosen as the design wave ?

Maximum wave: H_{max}, $T_{H_{max}}$

This is the wave which has the maximum wave height. In Table 1,

$$H_{max} = 5.5 \text{ m} \qquad\qquad T_{H_{max}} = 12.5 \text{ s}$$

Maximum wave is chosen as the design wave for structures which are very important and very sensitive to wave load, e.g. vertical breakwaters. Note H_{max} is a random variable with the distribution depending on the number of individual waves.

Highest one-tenth wave: $H_{1/10}$, $T_{H_{1/10}}$

$H_{1/10}$ is the average of the wave heights of the one-tenth highest waves. $T_{H_{1/10}}$ is the average of the wave periods associated with the one-tenth highest wave.

Significant wave: H_s, T_s or $H_{1/3}$, $T_{H_{1/3}}$

Significant wave height is the average of the wave heights of the one-third highest waves. Significant wave period is the average of the wave periods associated with the one-third highest wave. In Table 1,

$$H_s = \frac{1}{5}\sum_{i=1}^{5} H_i = 4.44 \text{ m} \qquad T_s = \frac{1}{5}\sum_{i=1}^{5} T_i = 12.8 \text{ s} \qquad i \text{ is the rank no.}$$

Significant wave is most often used as the design wave. The reason might be that in old days structures were designed based on visual observation of waves. Experiences show that the wave height and period reported by visual observation correspond approximately to significant wave. Therefore the choice of significant wave as design wave can make use of the existing engineering experience.

Mean wave: $\overline{H}$, $\overline{T}$

$\overline{H}$ and $\overline{T}$ are the means of the heights and periods of all individual waves. In Table 1,

$$\overline{H} = \frac{1}{15}\sum_{i=1}^{15} H_i = 2.9 \text{ m} \qquad\qquad \overline{T} = \frac{1}{15}\sum_{i=1}^{15} T_i = 9.25 \text{ s}$$

Root-mean-squre wave height H_{rms}

In Table 1,

$$H_{rms} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} H_i^2} = \sqrt{\frac{1}{15}\sum_{i=1}^{15} H_i^2} = 3.20 \text{ m}$$

Wave height with exceedence probability of $\alpha\%$: $H_{\alpha\%}$

For example $H_{0.1\%}$, $H_{1\%}$, $H_{2\%}$ etc.

1.3 Distribution of individual wave heights

Histogram of wave height

In stead of showing all individual wave heights, it is easier to use wave height histogram which tells the number of waves in various wave height intervals. Fig.3 is the histogram of wave height corresponding to Table 1.

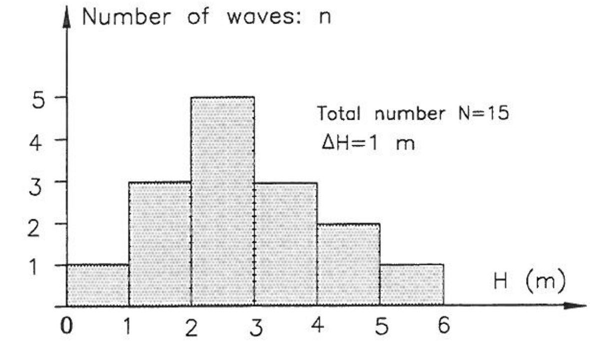


Fig.3. Histogram of wave height.

Non-dimensionalized histogram

In order to compare the distributions of wave height in different locations, the histogram of wave height is non-dimensionalized, cf. Fig.4.

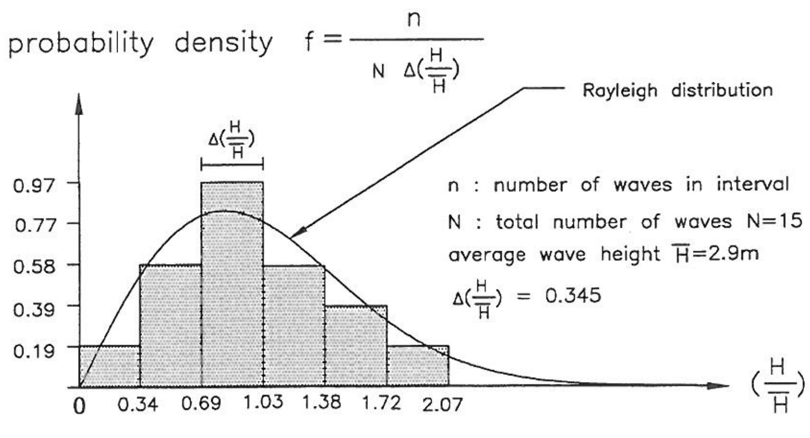


Fig.4. Non-dimensionalized histogram of wave height.

When $\Delta(H/\overline{H})$ approaches zero, the probability density becomes a continuous curve. Experience and theory have shown that this curve is very close to the Rayleigh distribution. Roughly speaking, we say that individual wave height follows the Rayleigh distribution.

Rayleigh distribution

The Rayleigh probability density function is

$$f(x) = \frac{\pi}{2} x \exp\left(-\frac{\pi}{4}x^2\right) \qquad x = \frac{H}{\overline{H}} \tag{1}$$

The Rayleigh distribution function is

$$F(x) = \text{Prob}\{X < x\} = 1 - \exp\left(-\frac{\pi}{4}x^2\right) \tag{2}$$

Relation between characteristic wave heights

If we adopt the Rayleigh distribution as an approximation to the distribution of individual wave heights, then the characteristic wave heights $H_{1/10}$, $H_{1/3}$, H_{rms} and $H_{\alpha\%}$ can be expressed by $\overline{H}$ through the manipulation of the Rayleigh probability density function.

$$\begin{aligned} H_{1/10} &= 2.03\,\overline{H} \\ H_{1/3} &= 1.60\,\overline{H} \\ H_{rms} &= 1.13\,\overline{H} \\ H_{2\%} &= 2.23\,\overline{H} \end{aligned} \tag{3}$$

Fig.5 illustrates how to obtain the relation between H_s and $\overline{H}$.

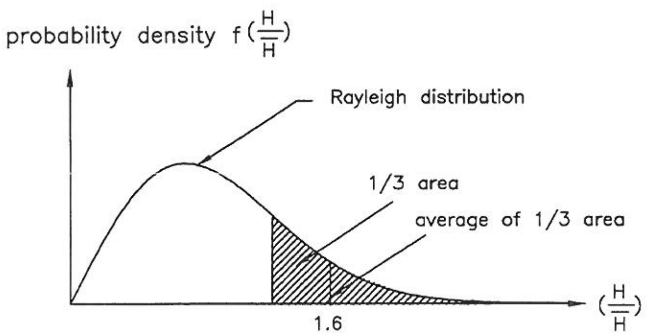


Fig.5. Relation between H_s and $\overline{H}$.

The Rayleigh distribution function given by H_s instead of $\overline{H}$ reads

$$F(H) = 1 - \exp\left(-2\left(\frac{H}{H_s}\right)^2\right) \tag{4}$$

Individual wave height distribution in shallow water

Only in relatively deep water, the Rayleigh distribution is a good approximation to the distribution of individual wave heights. When wave breaking takes place due to limited water depth, the individual wave height distribution will differ from the Rayleigh distribution.

Stive, 1986, proposed the following empirical correction to the Rayleigh distribution based on model tests but roughly checked against some prototype data

$$
\begin{aligned}
H_{1\%} &= H_{m_0}\left(\frac{ln100}{2}\right)^{\frac{1}{2}}\left(1+\frac{H_{m_0}}{h}\right)^{-\frac{1}{3}} \\
H_{0.1\%} &= H_{m_0}\left(\frac{ln1000}{2}\right)^{\frac{1}{2}}\left(1+\frac{H_{m_0}}{h}\right)^{-\frac{1}{2}}
\end{aligned} \tag{5}
$$

where h is the water depth, $H_{1\%}$ means the 1% exceedence value of the wave height determined by zero downcrossing analysis, whereas the significant wave height H_{m_0} is determined from the spectrum. The correction formulae are very useful for checking the wave height distribution in small scale physical model tests, cf. Fig.6.

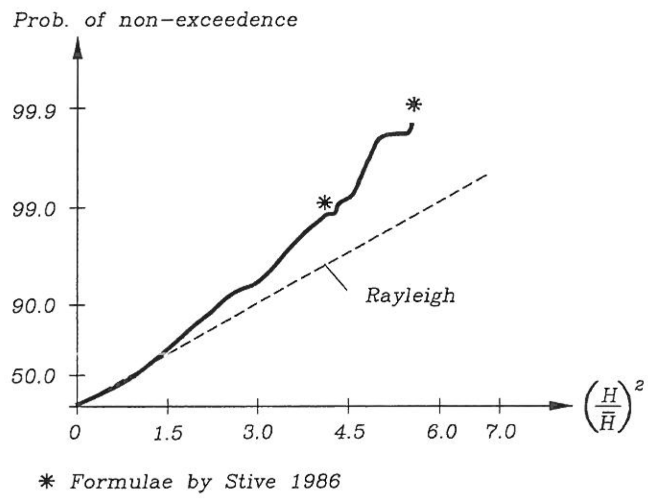


Fig.6. Comparison of the expression by Stive, 1986, for shallow water wave height distribution with model test results. Aalborg University Hydraulics Laboratory 1990 (from Burcharth 1993).

Klopmann et al. (1989) proposed a semi-empirical expression for the individual wave height distribution. Researches have also been done by Thornton and Guza (1983). Chapter 2 gives a more detailed discussion on the validity of the Rayleigh distribution, based on energy spectrum width parameter.

1.4 Maximum wave height H_{max}

The basic nature of H_{max} is that it cannot be calculated deterministically

<u>Distribution of H_{max}</u>

The distribution function of $X = H/\overline{H}$ is the Rayleigh distribution

$$F_X(x) \;=\; \text{Prob}\{X < x\} \;=\; 1 \;-\; \exp\left(-\frac{\pi}{4}x^2\right) \tag{6}$$

If there are N individual waves in a storm[1], the distribution function of $X_{max} = H_{max}/\overline{H}$ is

$$F_{X_{max}}(x) \;=\; \text{Prob}\{X_{max} < x\} \;=\; (\; F_X(x)\;)^N$$

$$=\; \left(1 \;-\; \exp\left(-\frac{\pi}{4}x^2\right)\right)^N \tag{7}$$

Note that $F_{X_{max}}(x)$ can be interpreted as the probability of the non-occurrence of the event ($X > x$) in any of N independent trials. The probability density function of X_{max} is

$$f_{X_{max}}(x) \;=\; \frac{\mathrm{d}F_{X_{max}}}{\mathrm{d}x}$$

$$=\; \frac{\pi}{2}\,N\,x\,\exp\left(-\frac{\pi}{4}x^2\right)\left(1 \;-\; \exp\left(-\frac{\pi}{4}x^2\right)\right)^{N-1} \tag{8}$$

The density function of X and the density function of X_{max} are sketched in Fig.7.

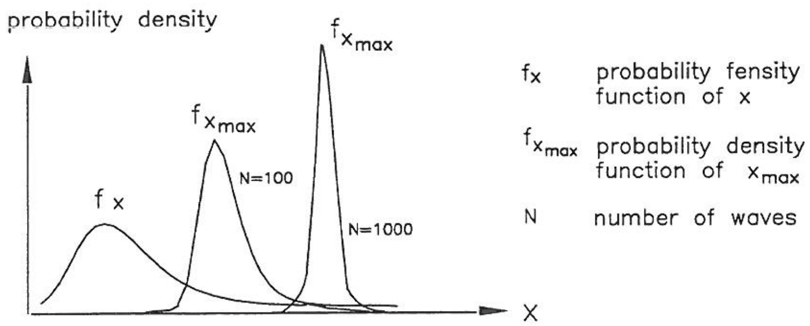
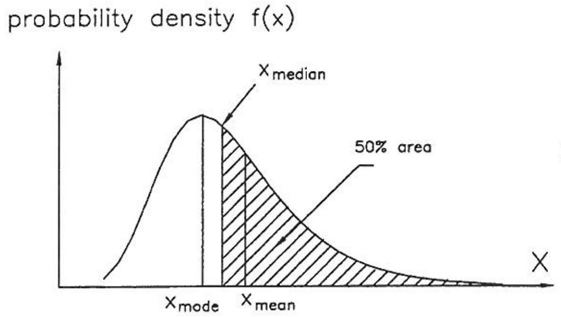


Fig. 7. Probability density function of X and X_{max}.

[1]A storm usually lasts some days. The significant wave height is varying during a storm. However we are more interested in the maximum significant wave height in a short period of time. In practice, N is often assumed to be 1000.

Mean, median and mode of H_{max}

Mean, median and mode are often used as the characteristic values of a random variable. Their definitions are given in Fig.8.



$$\text{Mean} \quad x_{mean} = E[X] = \int_{-\infty}^{+\infty} x f_X(x)\mathrm{d}x$$

$$\text{Median} \quad x_{median} = x\Big|_{F_X(x)=0.5}$$

$$\text{Mode} \quad x_{mode} = x\Big|_{f_X(x)=max}$$

Fig. 8. Mean, median and mode of a random variable X.

By putting eqs (7) and (8) into the definitions, we obtain

$$(H_{max})_{mean} \approx \left(\sqrt{\frac{\ln N}{2}} + \frac{0.577}{\sqrt{8 \ln N}} \right) H_s \tag{9}$$

$$(H_{max})_{mode} \approx \sqrt{\frac{\ln N}{2}}\, H_s \tag{10}$$

Furthermore, $(H_{max})_\mu$, defined as the maximum wave height with exceedence probability of μ (cf. Fig. 9), is

$$(H_{max})_\mu \approx \sqrt{\frac{1}{2} \ln \left(\frac{N}{\ln \left(\frac{1}{1-\mu} \right)} \right)} \tag{11}$$

Obviously $(H_{max})_{median} = (H_{max})_{0.5}$.

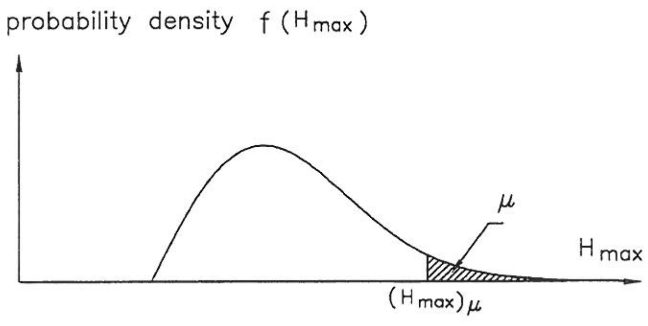


Fig. 9. Definition of $(H_{max})_\mu$.

Monte-Carlo simulation of H_{max} distribution

The distribution of H_{max} can also be studied by the Monte-Carlo simulation. Individual wave heights follow the Rayleigh distribution

$$F(H) = 1 - exp\left(-2 \left(\frac{H}{H_s}\right)^2 \right) \tag{12}$$

The storm duration corresponds to N individual waves.

1) Generate randomly a data between 0 and 1. Let the non-exceedence probability $F(H)$ equal to that data. One individual wave height H is obtained by (cf. Fig.10)

$$H = F^{-1}(F(H)) = H_s \sqrt{\frac{-\ln(1-F(H))}{2}} \tag{13}$$

2) Repeat step 1) N times. Thus we obtain a sample belonging to the distribution of eq (12) and the sample size is N.

3) Pick up H_{max} from the sample.

4) Repeat steps 2) and 3), say, 10,000 times. Thus we get 10,000 values of H_{max}.

5) Draw the probability density of H_{max}.

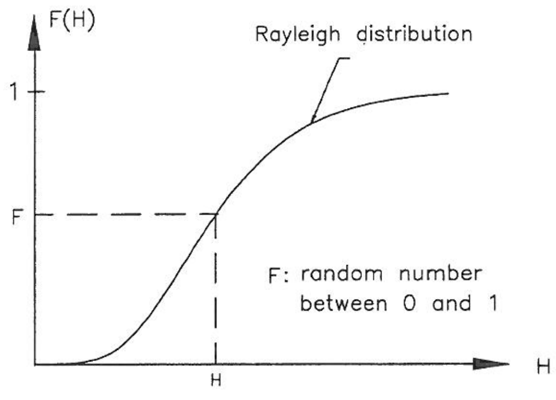


Fig.10. Simulated wave height from the Rayleigh distribution.

1.5 Distribution of wave period

It is summarized as

- There is no generally accepted expression for the distribution of wave period.
- The distribution of wave period is narrower than that of wave height.
- In practice the joint distribution of wave height and wave period is of great importance. Unfortunately, Until now there is no generally accepted expression for the joint distribution, even though there are some so-called *scatter diagrams* based on wave recording. Such a diagram is valid only for the measurement location. An example of scatter diagrams is given in Chapter 4, section 12. The relation between H_s and T_s is often simplified as $T_s = \alpha H_s^{\beta}$, e.g. in Canadian Atlantic waters $\alpha = 4.43$ and $\beta = 0.5$ (Neu 1982).
- The empirical relation $T_{max} \approx T_{1/10} \approx T_{1/3} \approx 1.2\,\overline{T}$ (Goda 1985).

1.6 References

Burcharth H.F. , 1993. *The design of breakwater.* Department of Civil Engineering, Aalborg University, 1993.

Dean, R.G. and Dalrymple, R.A. , 1991. *Water wave mechanics for Engineers and scientists.* Second printing with correction, World Scientific Publishing Co. Pte. Ltd., Singapore, 1991.

Goda, Y. , 1985. *Random seas and design of marine structures.* University of Tokyo Press, Japan, 1985

IAHR. , 1986. *List of sea state parameters.* Supplement to Bulletin No. 52, 1986

Klopmann, G. and Stive, M.J.F. , 1989. *Extreme waves and wave loading in shallow water.* E & P Forum, Report No. 3.12/156, 1989.

Neu, H.J.A. , 1982. *11-year deep water wave climate of Canadian Atlantic waters.* Canadian Tech. Rept. of Hydrography and Ocean Sciences, 13.

Stive, M.J.F. , 1986. *Extreme shallow water conditions.* Delft Hydraulics, Intern Report H533, 1986.

1.7 Exercise

1) The application of the down-crossing method gives the following 21 individual waves.

wave number	wave height H (m)	wave period T (s)
1	0.54	4.2
2	2.05	8.0
3	4.52	6.9
4	2.58	11.9
5	3.20	7.3
6	1.87	5.4
7	1.90	4.4
8	1.00	5.2
9	2.05	6.3
10	2.37	4.3

wave number	wave height H (m)	wave period T (s)
11	1.03	6.1
12	1.95	8.0
13	1.97	7.6
14	1.62	7.0
15	4.08	8.2
16	4.89	8.0
17	2.43	9.0
18	2.83	9.2
19	2.94	7.0
20	2.23	5.3
21	2.98	6.9

Calculate H_{max}, T_{max}, $H_{1/10}$, $T_{1/10}$, $H_{1/3}$, $T_{1/3}$, $\overline{H}$, $\overline{T}$, H_{rms}

2) Prove $H_{2\%} = 2.23\ \overline{H}$

3) Explain the difference between $H_{1/10}$ and $H_{10\%}$.

4) Suppose individual waves follow the Rayleigh distribution. Calculate the exceedence probability of $H_{1/10}$, H_s and $\overline{H}$.

5) An important coastal structure is to be designed according to H_{max}. The significant wave height of the design storm is $H_{1/3} = 10\ m$. The duration of the storm corresponds to 1000 individual waves.

(1) Calculate $(H_{max})_{mean}$, $(H_{max})_{mode}$, $(H_{max})_{median}$, $(H_{max})_{0.05}$

(2) Now suppose that the storm contains 500 individual waves. Calculate $(H_{max})_{mean}$, $(H_{max})_{mode}$, $(H_{max})_{median}$, $(H_{max})_{0.05}$. Compare with the results of (1).

(3) Use Monte-Carlo simulation to determine $(H_{max})_{mean}$, $(H_{max})_{mode}$, $(H_{max})_{median}$, $(H_{max})_{0.05}$

2 Time-series analysis II: Frequency-domain

The concept of spectrum can be attributed to Newton, who discovered that sunlight can be decomposed into a spectrum of colors from red to violet, based on the principle that white light consists of numerous components of light of various colors (wave length or wave frequency).

Energy spectrum means the energy distribution over frequency. Spectral analysis is a technique of decomposing a complex physical phenomenon into individual components with respect to frequency.

Spectral analysis of irregular waves is very important for structure design. For example, in the oil-drilling platform design where wave force plays an important role, it is of importance to design the structure in such a way that the natural frequency of the structure is fairly far away from the frequency band where most wave energy concentrates, so that resonance phenomenon and the resulted dynamic amplification of force and deformation can be avoided.

2.1 Some basic concepts of linear wave theory

Surface elevation

The surface elevation of a linear wave is

$$\eta(x,t) = \frac{H}{2}\cos(\omega t - kx + \delta) = a\ \cos(\omega t - kx + \delta) \tag{1}$$

where	H	wave height
	a	amplitude, $a = H/2$
	ω	angular frequency, $\omega = 2\pi/T$
	T	wave period.
	k	wave number, $k = 2\pi/L$
	L	wave length
	δ	initial phase

We can also define the observation location to $x = 0$ and obtain

$$\eta(t) = a\ \cos(\omega t + \delta) \tag{2}$$

The relation between wave period and wave length (dispersion relationship) is

$$L = \frac{g\ T^2}{2\pi}\ \tanh\left(\frac{2\pi h}{L}\right) \tag{3}$$

where h is water depth.

Wave energy

The average wave energy per unit area is

$$E = \frac{1}{8}\rho \; g \; H^2 = \frac{1}{2}\rho \; g \; a^2 \quad (\text{Joule/m}^2 \text{ in SI unit}) \tag{4}$$

Variance of surface elevation of a linear wave

The variance of the surface elevation of a linear wave is

$$\begin{aligned} \sigma_\eta^2 = \mathrm{Var}[\eta(t)] &= E\left[\left(\eta(t) - \overline{\eta(t)}\right)^2\right] && \text{(E: Expectation)} \\ &= E\;[\;\eta^2(t)\;] \\ &= \tfrac{1}{T}\int_0^T \eta^2(t)\,\mathrm{dt} && \text{(T: wave period)} \\ &= \tfrac{1}{2}\,a^2 \end{aligned}$$

Superposition of linear waves

Since the governing equation (Laplace equation) and boundary conditions are linear in small amplitude wave theory, it is known from mathematics that small amplitude waves are superposable. This means that the superposition of a number of linear waves with different wave height and wave period will be

	superposition		wave 1		wave 2		$\cdots$		wave N
velocity potential	φ	=	φ_1	+	φ_2	+	$\cdots$	+	φ_N
surface elevation	η	=	η_1	+	η_2	+	$\cdots$	+	η_N
particle velocity	u	=	u_1	+	u_2	+	$\cdots$	+	u_N
dynamic pressure	p	=	p_1	+	p_2	+	$\cdots$	+	p_N

2.2 Example of variance spectrum

First we will make use of an example to demonstrate what a variance spectrum is.

Surface elevation of irregular wave

Fig.1 gives an example of an irregular wave surface elevation which is constructed by adding 4 linear waves (component waves) of different wave height and wave period. The superposed wave surface elevation is

$$\eta(t) = \sum_{i=1}^{4} \eta_i(t) = \sum_{i=1}^{4} a_i \cos(\omega_i t + \delta_i) \tag{5}$$

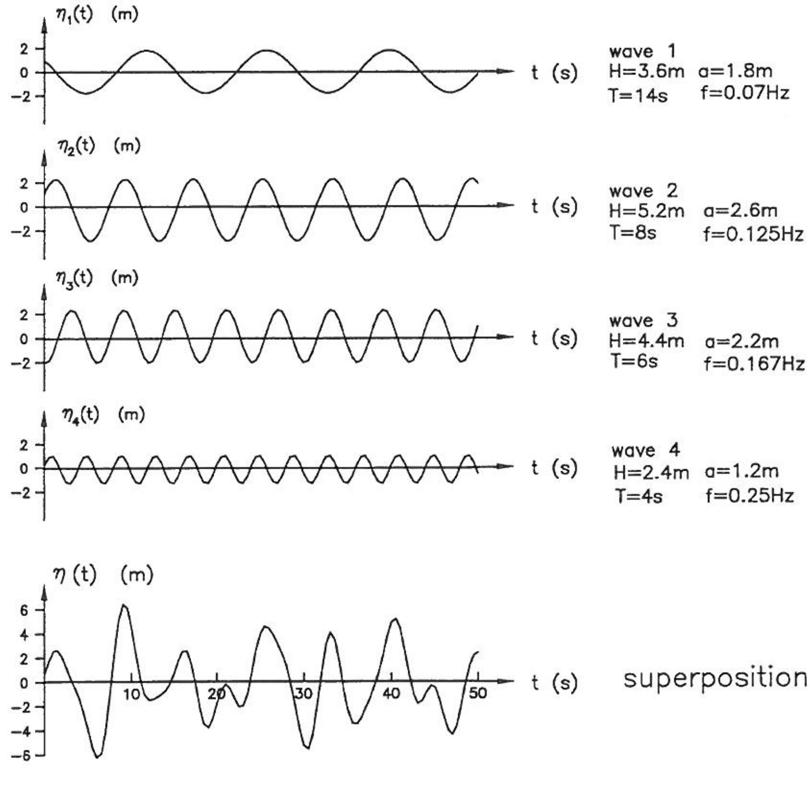


Fig.1. Simulation of irregular waves by superposition of linear waves.

Variance diagram

In stead of Fig.1, we can use a variance diagram, shown in Fig.2, to describe the irregular wave.

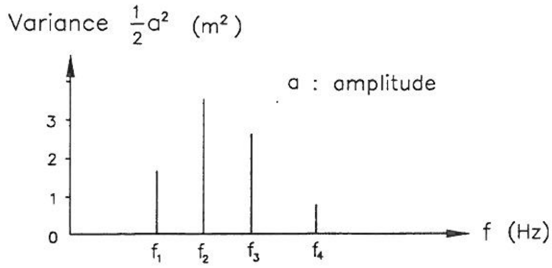


Fig.2. Variance diagram.

In comparison with Fig.1, the variance diagram keeps the information on amplitude (a_i) and frequency (f_i, hence T_i and L_i) of each component, while the information on initial phase (δ_i) is lost. This information loss does not matter because the surface elevation of irregular wave is a random process. We can simply assign a random initial phase to each component.

Variance spectral density $S_\eta(f)$

The variance diagram can be converted to variance spectrum, The spectral density is defined as

$$S_\eta(f) = \frac{\frac{1}{2}a^2}{\Delta f} \qquad (\mathrm{m}^2 \mathrm{~s}) \tag{6}$$

where Δf is the frequency band width[1], cf. Fig.3.

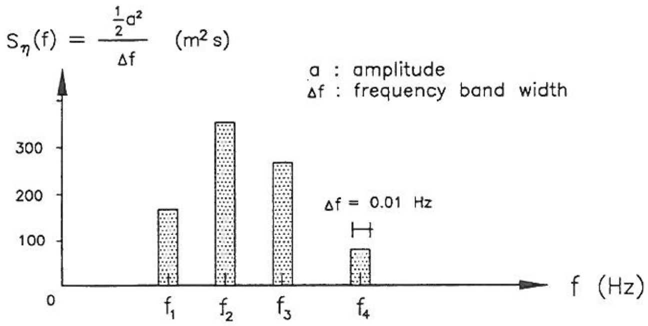


Fig.3. Stepped variance spectrum.

[1]we will see later that Δf depends on signal recording duration. In the figure it is assumed that $\Delta f = 0.01Hz$

In reality an irregular wave is composed of infinite number of linear waves with different frequency. Fig.4 gives an example of stepped variance spectrum. When Δf approaches zero, the variance spectrum becomes a continuous curve.

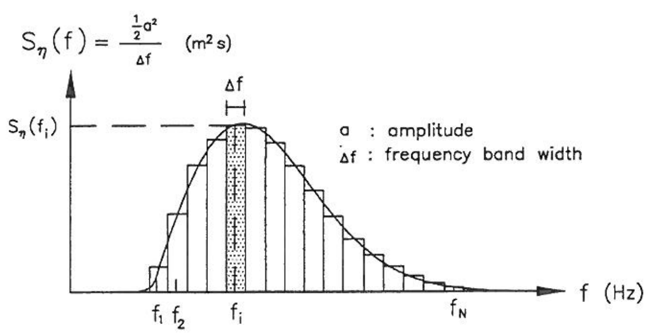


Fig.4. Continuous variance spectrum (wave energy spectrum).

Variance spectrum is also called energy spectrum. But strictly speaking, the energy spectral density should be defined as

$$S(f) = \frac{\frac{1}{2}\rho\ g\ a^2}{\Delta f} \qquad (\mathrm{m}^2\ \mathrm{s}) \tag{7}$$

<u>Construction of time series from variance spectrum</u>

We can also construct time series of surface elevation from variance spectrum. In fig.4 the known variance spectral density $S_\eta(f)$ is divided into N parts by the frequency band width Δf. This means that the irregular wave is composed of N linear waves

$$\eta(t) = \sum_{i=1}^{N} \eta_i(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \delta_i) \tag{8}$$

The variance of each linear wave is

$$S_\eta(f_i)\ \Delta f = \frac{1}{2} a_i^2 \qquad i = 1, 2, \cdots, N \tag{9}$$

Therefore the amplitude is

$$a_i = \sqrt{2\ S_\eta(f_i)\ \Delta f} \qquad i = 1, 2, \cdots, N \tag{10}$$

The angular frequency is

$$\omega_i = \frac{2\pi}{T_i} = 2\pi f_i \qquad i = 1, 2, \cdots, N \tag{11}$$

The initial phase δ_i is assigned a random number between 0 and 2π. Hence by use of eq (8) we can draw the time-series of the surface elevation of the irregular wave which has the variance spectrum as shown in Fig.4.

2.3 Fourier series

Conversion of irregular surface elevation into variance spectrum is not as simple as the above example, where the linear components of the irregular wave are pre-defined (cf. Fig.1). We need to decompose the irregular wave into its linear components. First let's see how it can be done with a known continuous function $x(t)$.

Fourier series

Fourier series is used to represent any arbitrary function[2].

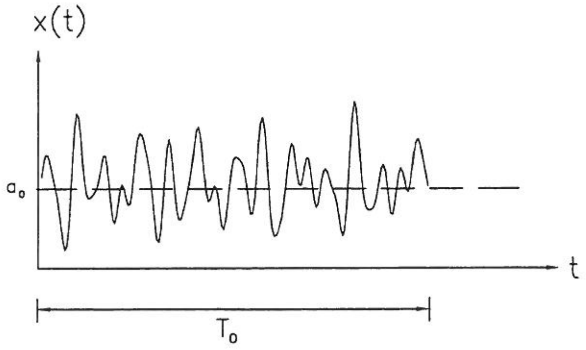


Fig.5. Arbitrary periodic function of time.

$$
\begin{aligned}
x(t) &= a_0 + 2\sum_{i=1}^{\infty}\left(a_i \cos\left(\frac{2\pi i}{T_0}t\right) + b_i \sin\left(\frac{2\pi i}{T_0}t\right)\right) \\
&= 2\sum_{i=0}^{\infty}\left(a_i \cos\omega_i t + b_i \sin\omega_i t\right)
\end{aligned} \tag{12}
$$

where a_i and b_i are Fourier coefficients given by

$$
\left.\begin{aligned}
a_i &= \tfrac{1}{T_0}\int_0^{T_0} x(t)\ \cos\omega_i t\ \mathrm{dt} \\
b_i &= \tfrac{1}{T_0}\int_0^{T_0} x(t)\ \sin\omega_i t\ \mathrm{dt}
\end{aligned}\right\} \quad i = 0, 1, 2, \cdots, \infty \tag{13}
$$

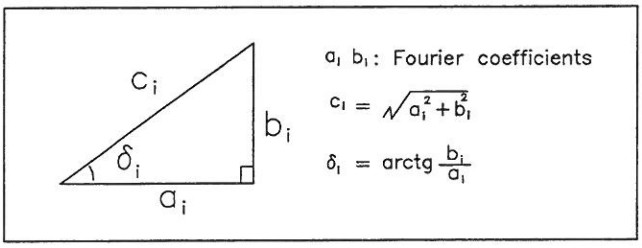
Note $a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\mathrm{dt}$ and $b_0 = 0$.

[2]Not all mathematicians agree that an arbitrary function can be represented by a Fourier series. However, all agree that if $x(t)$ is a periodic function of time t, with period T_0 then $x(t)$ can be expressed as a Fourier series. In our case $x(t)$ is the surface elevation of irregular wave, which is a random process. if T_0 is large enough, we can assume that $x(t)$ is a periodic function with period T_0.

Physical interpretation

Now we say that the continuous function $x(t)$ is the surface elevation of irregular wave $\eta(t)$, which can be expanded as a Fourier series.

$$\eta(t) \;=\; 2\sum_{i=0}^{\infty}(a_i \cos\omega_i t \;+\; b_i \sin\omega_i t)$$



$$\begin{aligned}
&= 2\sum_{i=0}^{\infty}(c_i \cos\delta_i \;\cos\omega_i t \;+\; c_i \sin\delta_i \;\sin\omega_i t) \\
&= \sum_{i=0}^{\infty} 2c_i(\cos\delta_i \;\cos\omega_i t \;+\; \sin\delta_i \;\sin\omega_i t) \\
&= \sum_{i=0}^{\infty} 2c_i \;\cos(\omega_i t \;-\; \delta_i\,) \qquad (14)
\end{aligned}$$

That is to say, any irregular wave surface elevation, expressed as a continues function, is composed of infinite number of linear waves with

$$\left.\begin{array}{lll} \text{amplitude} & 2c_i \;=\; 2\sqrt{a_i^2 + b_i^2} \\ \text{period} & T_i \;=\; \frac{2\pi}{\omega_i} \;=\; \frac{T_0}{i} \end{array}\right\} \quad i = 0, 1, \cdots, \infty \qquad (15)$$

$\{a_i,\; b_i\}$, $i = 0, 1, 2, \cdots, \infty$, are given in eq (13).

2.4 Discrete signal analysis

The measurement of surface elevation is carried out digitally. We do not have, neither necessary, a continuous function of the surface elevation. In stead we have a series of surface elevation measurement equally spaced in time, cf. Fig.6.

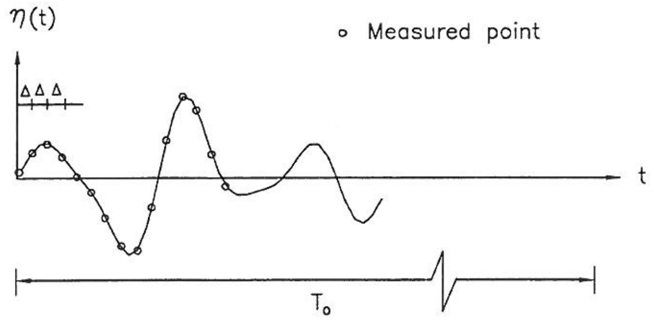


Fig.6. Sampling of surface elevation at regular intervals.

If the sampling frequency is f_s, then the time interval between two succeeding points is $\Delta = 1/f_s$. Corresponding to the total number of sample points N, the sample duration $T_0 = (N-1)\Delta$. Thus we obtain a discrete time series of surface elevation

$$\eta_0, \quad \eta_1, \quad \cdots, \quad \eta_{N-1}$$

The Fourier coefficients

$$(a_0,\ b_0), \quad (a_1,\ b_1), \quad \cdots, \quad (a_{N-1},\ b_{N-1})$$

can be obtained by Fast Fourier Transforms (FFT)[3]. That is to say, the irregular wave surface elevation, expressed by digital time series, is composed of N linear waves

$$\eta(t) = \sum_{i=0}^{N-1} \eta_i(t) = \sum_{i=0}^{N-1} 2\sqrt{a_i^2 + b_i^2}\ \cos(\omega_i t + \delta_i) \tag{16}$$

$$\left.\begin{array}{ll} \text{amplitude} & 2\sqrt{a_i^2 + b_i^2} \\ \text{angular frequency} & \omega_i = \frac{2\pi i}{T_0} \\ \text{period} & T_i = \frac{2\pi}{\omega_i} = \frac{T_0}{i} \\ \text{frequency} & f_i = \frac{1}{T_i} = \frac{i}{T_0} \end{array}\right\} \quad i = 0, 1, \cdots, N-1 \tag{17}$$

[3]FFT is a computer algorithm for calculating DFT. It offers an enormous reduction in computer processing time. For details of DFT and FFT, please refer to Newland (1975)

Therefore we obtain the variance spectrum

$$\text{frequency band width} \qquad \Delta f \;=\; f_{i+1} - f_i \;=\; \frac{1}{T_0}$$

$$\text{spectral density} \qquad S_\eta(f_i) \;=\; \frac{\frac{1}{2}(amplitude)^2}{\Delta f} \;=\; \frac{2(a_i^2 + b_i^2)}{\Delta f} \tag{18}$$

An example of variance spectrum is shown in Fig.7.

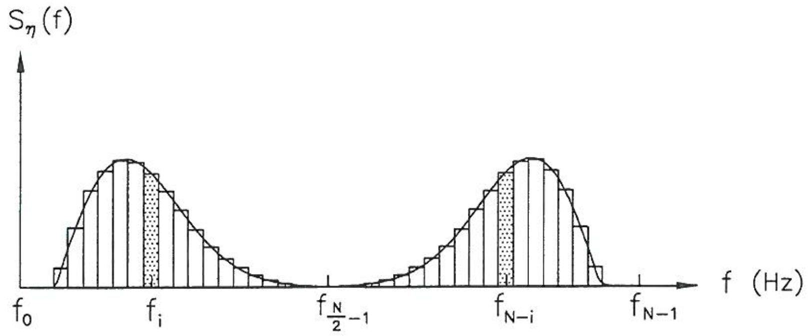


Fig.7. Variance spectrum.

Nyquist frequency $f_{nyquist}$

Nyquist frequency $f_{nyquist}$ is the maximum frequency which can be detected by the Fourier analysis.

Fourier analysis decomposes N digital data into N linear components. The frequency of each component is

$$f_i \;=\; \frac{i}{T_0} \qquad i = 0, 1, \cdots, N-1 \tag{19}$$

The nyquist frequency is

$$f_{nyquist} \;=\; f_{\frac{N-1}{2}} \;=\; \frac{\frac{N-1}{2}}{T_0} \;=\; \frac{\frac{N-1}{2}}{(N-1)\,\Delta} \;=\; \frac{1}{2\,\Delta} \;=\; \frac{f_s}{2} \tag{20}$$

where f_s sample frequency

Δ time interval between two succeeding sample points, $\Delta = 1/f_s$

N total number of sample

T_0 sample duration, $T_0 \;=\; (N-1)\,\Delta$

The concept of nyquist frequency means that the Fourier coefficients $\{\ a_i,\ b_i\ \}$, $i = 0, 1, \cdots, N-1$, contains two parts, the first half part ($i = 0, 1, \cdots, N/2-1$) represents true components while the second half part ($i = N/2, N/2+1, \cdots, N-1$) is the folding components (aliasing).

Fig.8 gives an example on aliasing after the Fourier analysis of discrete time series of a linear wave.

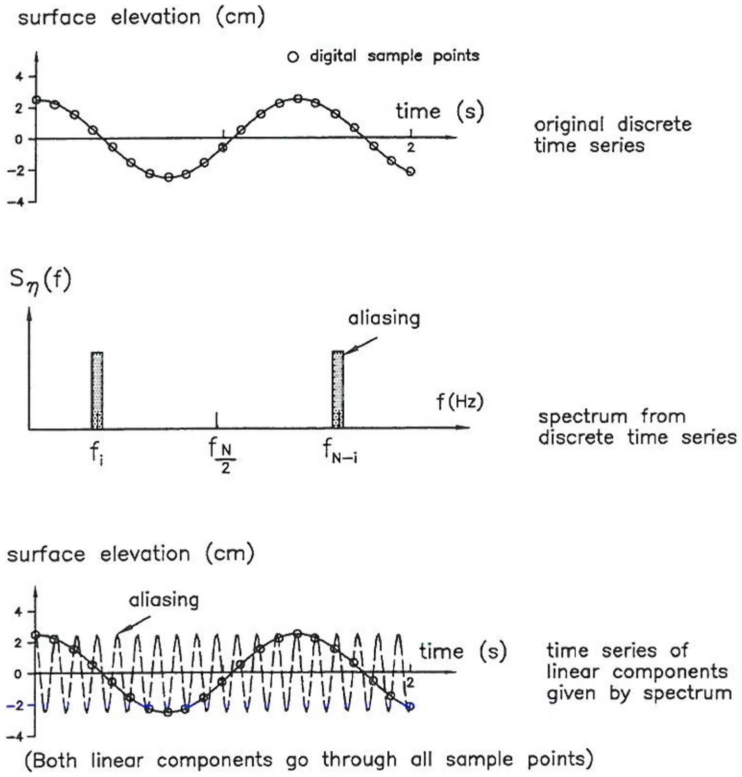


Fig.8. Aliasing after Fourier analysis.

The solution to aliasing is simple: let $\{ a_i, b_i \}$, $i = N/2, N/2+1, \cdots, N-1$, equal to zero, cf. Fig.9. That is the reason why $f_{nyquist}$ is also called cut-off frequency. In doing so we are actually assuming that irregular wave contains no linear components whose frequency is higher than $f_{nyquist}$. This assumption can be assured by choosing sufficiently high sample frequency f_s, cf. eq (20).

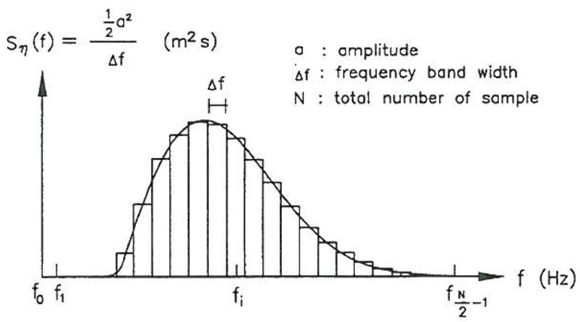


Fig.9. Variance spectrum after cut-off (refer to Fig.7).

Taper data window

Fourier analysis requires that $\eta(t)$ is a periodic function with period T_0, it may be desirable to modify the recorded time series before Fourier analysis, so that the signal looks like a periodic function. The modification is carried out with the help of taper data window.

The widely-used cosine taper data window reads

$$d(t) \;=\; \begin{cases} \frac{1}{2}\left(1-\cos\frac{10\pi t}{T_0}\right) & 0 \le t \le \frac{T_0}{10} \\ 1.004 & \frac{T_0}{10} \le t \le \frac{9T_0}{10} \\ \frac{1}{2}\left(1+\cos\frac{10\pi\left(t-\frac{9T_0}{10}\right)}{T_0}\right) & \frac{9T_0}{10} \le t \le T_0 \end{cases} \tag{21}$$

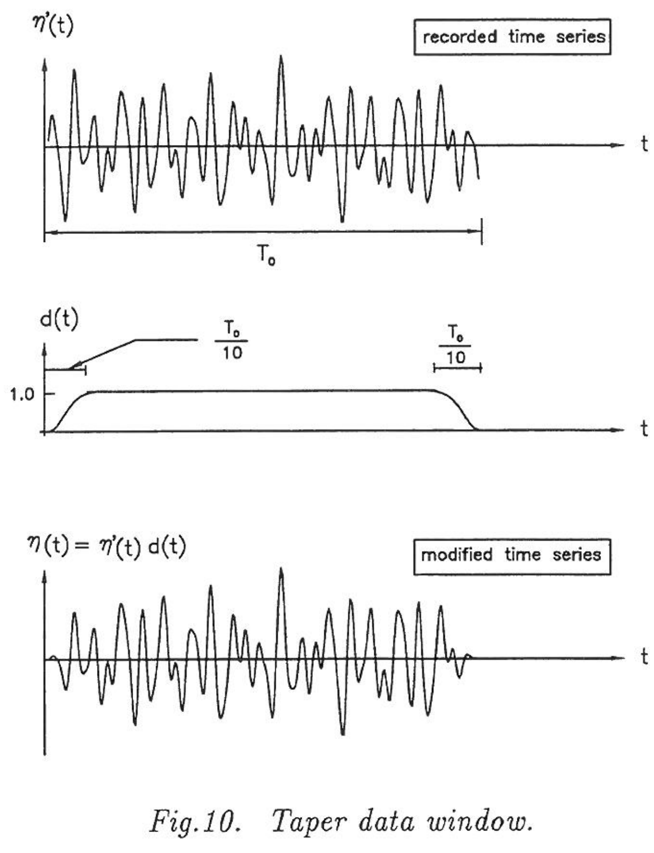


Fig.10. Taper data window.

2.5 Characteristic wave height and period

The variance spectrum, illustrated in Fig.11, says nothing about how high the individual waves will be. Now We will see how to estimate the characteristic wave height and period based on the variance spectrum.

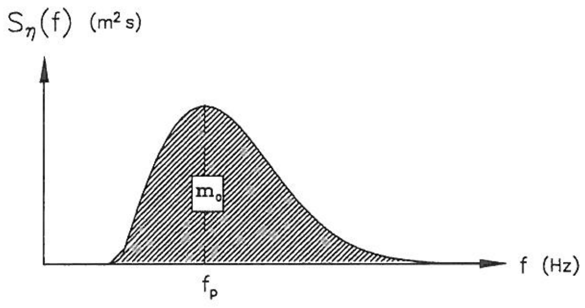


Fig.11. Variance spectrum.

<u>n order moment m_n</u>

m_n is defined as

$$m_n = \int_0^\infty f^n \, S_\eta(f) \, \mathrm{df} \tag{22}$$

The zero moment is

$$m_0 = \int_0^\infty S_\eta(f) \, \mathrm{df} \tag{23}$$

which is actually the area under the curve, cf. Fig.11.

<u>Spectrum width parameter and validity of the Rayleigh distribution</u>

From the definition of m_n, it can be seen that the higher the order of moment, the more weight is put on the higher frequency portion of the spectrum. With the same m_0, a wider spectrum gives larger values of the higher order moment ($n \geq 2$). Longuet-Higgins has defined a spectrum width parameter

$$\varepsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} \tag{24}$$

It has been proven theoretically that

spectrum width parameter		wave height distribution
$\varepsilon = 0$	narrow spectrum	Rayleigh distribution
$\varepsilon = 1$	wide spectrum	Normal distribution

In reality ε lies in the range of 0.4-0.5. It has been found that Rayleigh distribution is a very good approximation and furthermore conservative, as the Rayleigh distribution gives slightly larger wave height for any given probability level.

Significant wave height H_{m_0} and peak wave period T_p

When wave height follows the Rayleigh distribution, i.e. $\varepsilon = 0$, the significant wave height H_{m_0}[4] can theoretically be expressed as

$$H_{m_0} = 4\sqrt{m_0} \tag{25}$$

In reality where $\varepsilon = 0.4 - 0.5$, a good estimate of significant wave height from energy spectrum is

$$H_{m_0} = 3.7\sqrt{m_0} \tag{26}$$

Peak frequency is defined as (cf. Fig.11)

$$f_p = f\Big|_{S_\eta(f)=max} \tag{27}$$

Wave peak period ($T_p = 1/f_p$) is approximately equal to significant wave period defined in time-domain analysis.

2.6 References

Burcharth, H.F. and Brorsen, M. , 1978. *On the design of gravity structures using wave spectra.* Lecture on Offshore Engineering, Edited by W.J.Graff and P. Thoft-Christensen, Institute of Building Technology and Structural Engineering, Aalborg University, Denmark.

Goda, Y. , 1985. *Random seas and design of marine structures.* University of Tokyo Press, Japan, 1985

Newland, D.E. , 1975. *In introduction to random vibrations and spectral analysis.* Longman, London, 1975.

[4] H_{m_0} denotes significant wave height determined from spectrum while H_s or $H_{1/3}$ is significant wave height determined from time-domain analysis. They are equal to each other when wave height follows the Rayleigh distribution.

2.7 Exercise

1) An irregular wave is composed of 8 linear components with

wave no.	1	2	3	4	5	6	7	8
wave height H (m)	5.0	4.3	3.8	3.6	3.3	2.8	2.2	0.3
wave period T (s)	10.3	12	9.4	14	7	6.2	5	3.3

The recording length is 20 seconds. Draw the variance diagram and variance spectrum of the irregular wave.

2) Convert the variance spectrum obtained in exercise 1) into time series of surface elevation.

3) Make a computer program to simulate the surface elevation of an irregular wave which is composed of 8 linear components. Wave height and period of each component are given in exercise 1). Suppose the sample frequency is 3 Hz and the recording length is 500 seconds.

   (1) Determine H_s and T_s by time-domain analysis.

   (2) Compare the distribution of individual wave height with the Rayleigh distribution.

   (3) Calculate total number of linear components to be given by Fourier analysis N, frequency band width Δf, and the nyquist frequency $f_{nyquist}$.

   (4) Draw the variance spectrum of the irregular wave by FFT analysis. (only for those who have interest.)

4) In reality where $\varepsilon = 0.4 - 0.5$, a good estimate of significant wave height from energy spectrum is

$$H_{m_0} = 3.7\sqrt{m_0}$$

Try to find out the principle of getting this empirical relation.

3 Wind-generated waves

If a structure is to be built at the location where there is no direct wave measurement, wave characteristics may be estimated by wind data.

Two simplified methods have been used to determine wave characteristics from a known wind field. The one is called SPM-method (Shore Protection Manual, 1984), which is the modification of Sverdrup-Munk-Bretschneider method (SMB-method). SPM-method gives significant wave height (H_{m_0}) and peak period (T_p) in terms of wind field [1]. The other is called spectrum-method, which gives variance spectrum in terms of wind field. When required, a significant wave height and peak period can be estimated from the spectrum and the results will be the same as SPM-method.

Besides these two simplified method, there are also numerical methods solving a differential equation governing the growth of wave energy. This approach will not be discussed in detail because the application of such models require specialized expertise.

3.1 Wave development and decay

Wind waves grow as a result of a flux of energy from the air into the water. When the wind velocity near the water surface exceeds a critical value of about 1 m/s, one can observe water surface ripples of length 5-10 cm and height 1-2 cm.

The process of wave development is complex. First the wind-wave interaction transfers wind energy to shorter waves. Then the wave-wave interaction transfers energy in shorter waves to energy in longer waves, thus resulting in the growth of longer waves.

Wind energy can be transferred to the waves only when the component of surface wind in the direction of wave travel exceeds the speed of wave propagation. Waves begin to decay when winds decrease in intensity or change in direction, or waves propagate out of wind field.

Therefore a change in wave energy depends on the transformation of the wind's kinetic energy into the wave energy, the transformation of wave energy at one frequency into wave energy at other frequencies, the dissipation of wave energy into turbulence by friction, viscosity and breaking, the advection of wave energy into and out of a region.

[1] Notice that H_{m_0} is the significant wave height determined from variance spectrum. In Shore Protection Manual (1984) the peak period is denoted T_m.

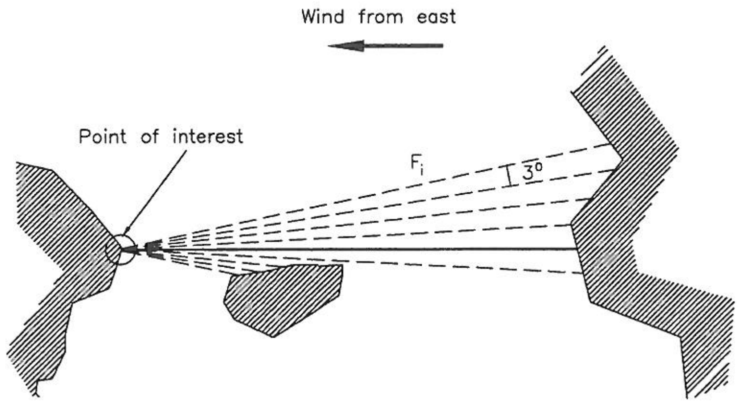
3.2 SPM-method

This method is presented in Shore Protection Manual (1984), edited by the US Army Corps of Engineers, Coastal Engineering Research Center (CERC).

Involved parameters

1 Fetch (F in (m)): Fetch is the distance between the point of interest and shoreline in the up-wind direction. Because the fetches surrounding the wind direction will influence the wind generated waves, SPM (1984) recommends to construct 9 radials from the point of interest at 3-degree intervals and to extend these radials until they first intersect shorelines. The fetch is equal to the average of the length of these 9 radials, i.e.

$$F = \frac{\sum_{i=1}^{9} F_i}{9} \tag{1}$$



2 Wind stress factor (U_A in (m/s)): Wind stress is most directly related to wave growth. The accurate estimation of vertical profile of wind speed, and hence wind stress, involves the air-sea temperature difference, sea surface roughness and friction velocity. In SPM (1984), all these factors are accounted for by using U_A

a. Elevation. If the given wind speed is not measured at the 10 meter elevation, the wind speed must be adjusted accordingly by

$$U_{10} = U_z \left(\frac{10}{z}\right)^{1/7} \quad \text{for } z < 20 \text{ m} \tag{2}$$

where U_{10} and U_z are wind speed at the elevation of 10 m and z m respectively.

b. Location effects. If wind speeds is estimated by visual observations on ships, they should be corrected by

$$U = 2.16\ U_s^{(\frac{7}{9})} \tag{3}$$

where U_s is the ship-reported wind speed in knots and U is the corrected wind speed in knots.
If wind data over water is not available, but data from nearby land site are, Fig.1 can be used to convert overland winds to overwater winds if they are the result of the same pressure gradient and the only major difference is the surface roughness

c. Stability correction. If the air-sea temperature difference ($\Delta T = T_{\mathrm{air}} - T_{\mathrm{sea}}$) is zero, the boundary layer is stable and wind speed correction is unnecessary. If ΔT is negative, the boundary layer is unstable and wind speed is more effective in causing wave growth. If ΔT is positive, the boundary layer is unstable and the wind speed is less effective. Fig.2 gives the wind speed amplification factor (R_T) due to air-sea temperature difference. In the absence of temperature information $R_T = 1.1$ can be applied.

d. Duration-averaged wind speed. The wind speed is often observed and reported as the maximum short-duration-averaged-speed. This should be converted to the wind speed averaged in an appropriate duration by

$$\frac{U_t}{U_{t=3600s}} = 1.277 + 0.296 \tanh\left(0.9 \log_{10}\left(\frac{45}{t}\right)\right) \quad \text{for } 1s < t < 3600s \tag{4}$$

$$\frac{U_t}{U_{t=3600s}} = 1.5334 - 0.15\ \log_{10}(t) \quad \text{for } 3600s < t < 36000s \tag{5}$$

where U_t is the average wind speed in t seconds.

e. Wind-stress factor. The wind-stress factor is implemented in order to account for the nonlinear relationship between wind stress and wind speed.

$$U_A = 0.71\ U_{10}^{1.23} \tag{6}$$

where U_{10} is the wind speed at the height of 10 m over mean water level, modified according to location and air-sea temperature, and averaged over an appropriate duration. It should be noted that the unit of U_{10} and U_A is m/s because eq (6) has no unit-homogenity.

3 Wind duration (t in (s)) and water depth (h in (m)).

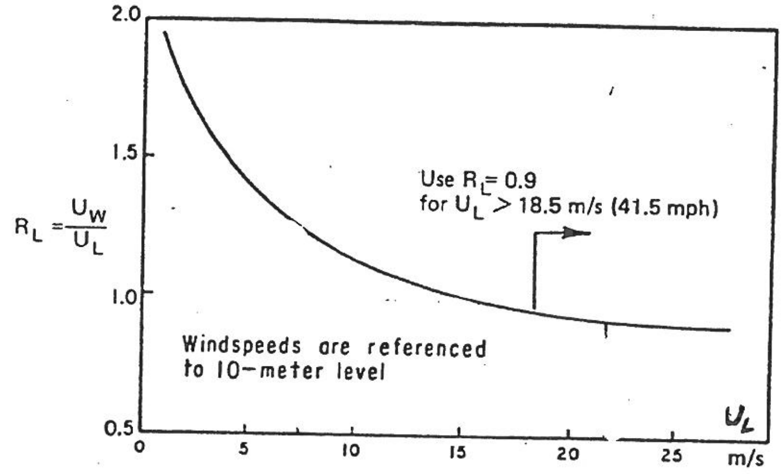


Fig.1. Ratio of wind speed over water (U_w) to wind speed over land (U_L) (scanned from SPM 1984).

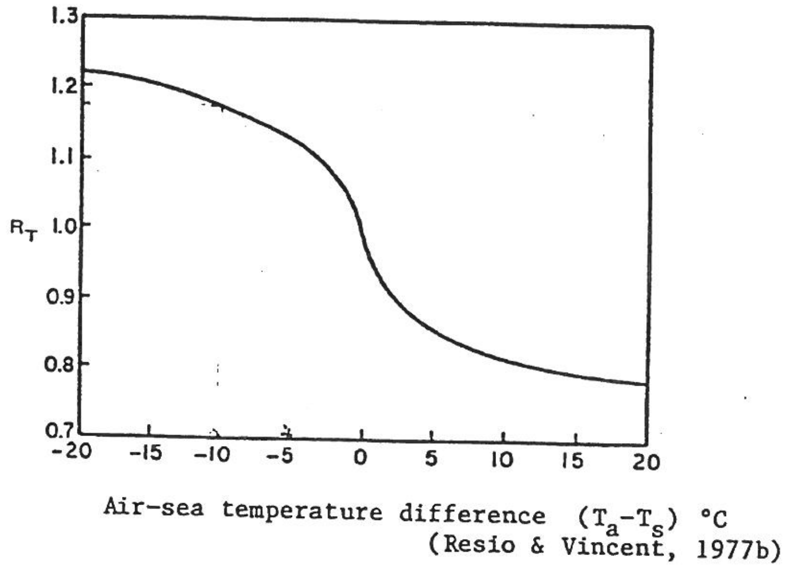


Fig.2. Amplification factor accounting for the effect of air-sea temperature difference (scanned from SPM 1984).

We may expresses the significant wave height and peak period in functional forms

$$H_{m_0},\ T_p \ = \ f\ (\ U_A,\ F,\ t,\ h\ ) \tag{7}$$

A dimensional analysis applied to eq (7) gives

$$\frac{g\ H_{m_0}}{U_A^2}\ ,\ \frac{g\ T_p}{U_A}\ =\ f\ \left(\frac{g\ F}{U_A^2}\ ,\ \frac{g\ t}{U_A}\ ,\ \frac{g\ h}{U_A^2}\right) \tag{8}$$

where $g = 9.81\ (m/s^2)$ is the gravitational acceleration.

<u>Fetch-limited case</u>

It is the situation where the wind has blown constantly long enough for wave heights at the end of the fetch to reach equilibrium.

1 Deep water ($\frac{h}{L} > \frac{1}{2}$): The condition for deep water waves to be fetch-limited is that the wind duration is longer than the minimum necessary duration t_{min}, given by

$$\frac{g\ t_{min}}{U_A}\ =\ 68.8\ \left(\frac{g\ F}{U_A^2}\right)^{2/3} \tag{9}$$

Significant wave height and peak period under fetch-limited condition are

$$\begin{cases} \frac{g\ H_{m0}}{U_A^2} & = \ 0.0016\ \left(\frac{g\ F}{U_A^2}\right)^{1/2} \\ \frac{g\ T_p}{U_A} & = \ 0.2857\ \left(\frac{g\ F}{U_A^2}\right)^{1/3} \end{cases} \tag{10}$$

Eq (10) shows, a larger fetch gives a larger wave height and longer wave period. But there is a limit, the so-called fully arisen sea. This wave condition refers to the case where the waves have reached an equilibrium state in which energy input from the wind is exactly balanced by energy loss. The fully arisen sea occurs when

$$\frac{g\ F}{U_A^2} \geq 23123 \tag{11}$$

That is to say, eq (10) is valid up to $\frac{g\ F}{U_A^2} = 23123$. When $\frac{g\ F}{U_A^2} > 23123$, waves become fully-arisen, and the significant wave height and peak period are

$$\begin{cases} \frac{g\ H_{m0}}{U_A^2} & = \ 0.0016\ (\ 23123\ )^{1/2}\ =\ 0.243 \\ \frac{g\ T_p}{U_A} & = \ 0.2857\ (\ 23123\ )^{1/3}\ =\ 8.134 \end{cases} \tag{12}$$

2 Transitional or shallow water ($\frac{h}{L} < \frac{1}{2}$): Waves feel the effect of sea bottom. Some part of wave energy dissipates due to bottom friction and percolation. For the same wind speed and fetch, wave height will be smaller and wave period shorter in comparison with deep water situation. SPM (1984) suggests the following formulae

$$\frac{g\ H_{m0}}{U_A^2} = 0.283\ \tanh\left(0.53\ \left(\frac{g\ h}{U_A^2}\right)^{3/4}\right)\ \tanh\left[\frac{0.00565\left(\frac{g\ F}{U_A^2}\right)^{1/2}}{\tanh\left(0.53\left(\frac{g\ h}{U_A^2}\right)^{3/4}\right)}\right]$$

$$\frac{g\ T_p}{U_A} = 7.54\ \tanh\left(0.833\ \left(\frac{g\ h}{U_A^2}\right)^{3/8}\right)\ \tanh\left[\frac{0.0379\left(\frac{g\ F}{U_A^2}\right)^{1/3}}{\tanh\left(0.833\left(\frac{g\ h}{U_A^2}\right)^{3/8}\right)}\right]$$

$$\frac{g\ t_{min}}{U_A} = 537\ \left(\frac{g\ T_p}{U_A}\right)^{7/3}$$

SPM (1984) calls the above formulae 'interim formulae' because the modification is ongoing in order to make the above formulae consistent with deep water.

Duration-limited

It is the situation where the wind duration is shorter than the minimum necessary duration.

There is no generally accepted formula. SPM (1984) suggests to make use of the formulae for the fetch-limited situation. It proceeds as

1) Check out $t < t_{min}$, i.e. duration limited
2) Replace t_{min} by t in eq (9) and calculate the fictional fetch F
3) Calculate H_{m0} and T_p by eq (10) where the fetch is the fictional fetch.

Example Application of SPM-method

Given Eight consecutive hourly observations of fastest mile wind speed $U_0 = 20\ m/s$ are observed at an elevation $z = 6\ m$, approximately 5 kilometers inland from shore. The observation site is at an airport weather station. The air-sea temperature difference is estimated to be -6^0C.

Wanted H_{m_0} and T_p for the fetch 100 kilometers at a deep water location.

Solution Fastest mile wind speed is the fastest wind speed, averaged over the duration equal to the time needed for the fastest wind speed to travel 1 mile. 1 mile = 1609 m.

We proceed as follows

1. Elevation adjustment

$$U_{10} = U_z \left(\frac{10}{z}\right)^{1/7} = U_0 \left(\frac{10}{6}\right)^{1/7} = 21.5\ m/s$$

2. Location adjustment

From Fig.1 it is found $R_L = 0.9$, the wind speed is adjusted to $0.9 \times 21.5 = 19.4\ m/s$.

3. Temperature adjustment

From Fig.2 it is found $R_T = 1.14$, the wind speed is adjusted to $1.14 \times 19.4 = 22.1\ m/s$.

4. Duration adjustment

The duration over which the fastest mile wind speed is averaged is actually the time needed for the fastest mile wind speed to travel one mile.

$$t = \frac{1609}{22.1} = 72.8\ s$$

i.e. the wind velocity of 22.1 m/s is the average velocity in 72.8 s, denoted as $U_{t=72.8}$. It should be converted to the average wind velocity in one hour, because in this example the fastest mile wind speed is given on hourly basis.

$$\frac{U_{t=72.8}}{U_{t=3600}} = 1.277 + 0.296 \tanh\left(0.9 \log_{10}\left(\frac{45}{72.8}\right)\right) = 1.22$$

$$U_{t=3600} = U_{t=72.8}/1.22 = 18.1\ m/s$$

5. The wind stress factor is

$$U_A = 0.71\ (18.1)^{1.23} = 25\ m/s$$

6. Type of wind wave

The given fastest wind speed indicates that wind is constant in 8 hours, the minimum necessary wind duration is

$$t_{min} = 68.8 \left(\frac{g\,F}{U_A^2}\right)^{2/3} \left(\frac{U_A}{g}\right) = 23688\ s = 6.6 \text{ hours} < 8 \text{ hours}$$

Therefore it is fetch-limited condition.

Because

$$\frac{g\,F}{U_A^2} = 1568 < 23123$$

it is not fully arisen sea.

7. H_{m_0} and T_p are given by

$$H_{m_0} = 0.0016 \left(\frac{g\,F}{U_A^2}\right)^{1/2} \left(\frac{U_A^2}{g}\right) = 4.04\ m$$

$$T_p = 0.2857 \left(\frac{g\,F}{U_A^2}\right)^{1/3} \left(\frac{U_A}{g}\right) = 8.47\ s$$

3.3 Standard variance spectrum

PM spectrum

In 1964, W.J.Pierson and L.Moskowitz put forward, on the basis of a similarity theory by S.A.Kitaigorodskii, some suggestions for deep water wave spectra for the sea state referred to as *fully arisen sea*. This wave condition refers to the case where the waves have reached an equilibrium state in which energy input from the wind is exactly balanced by energy loss. The only variable is thus the wind velocity. It is important to emphasize, that spectra of this type are only valid when the fetches are large enough to reach this equilibrium.

Out of the three analytical expressions suggested by Pierson and Moskowitz, the one below was found to give the best agreement with empirical wave data. This spectrum is called PM spectrum.

$$S_\eta(f) = \frac{\alpha g^2}{(2\pi)^4} f^{-5}\, exp\left(-0.74\left(\frac{f_0}{f}\right)^4\right) \tag{13}$$

$$\alpha = 0.0081$$
$$f_0 = g\,(2\pi U_{19.5})^{-1}$$

$U_{19.5}$: Wind speed, 19.5 m above mean water level

g : Gravitational acceleration

PM spectrum has been transformed to parameterized spectrum by $H_s = 4\,\sqrt{m_0}$ and $T_p = 1.4\,\overline{T} = 1.4\,\frac{m_0}{m_1}$

$$S_\eta(f) = \frac{5}{16} H_s^2\, f_p^4\, f^{-5}\, exp\left(-\frac{5}{4}\left(\frac{f_p}{f}\right)^4\right) \tag{14}$$

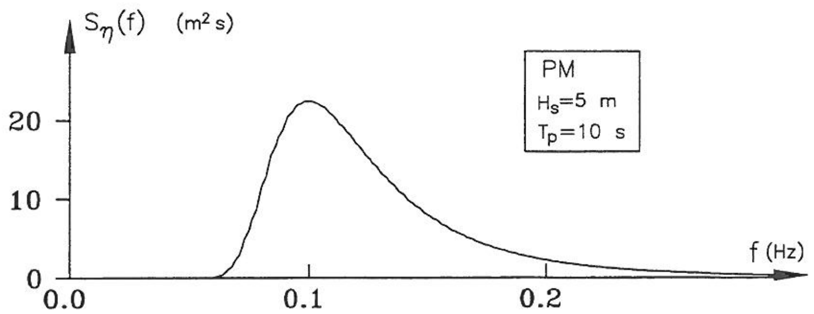


Fig.3. Example of PM spectrum.

JONSWAP spectrum

The Joint North Sea Wave Project (JONSWAP) was started in 1967 as a collaboration among institutes in Germany, Holland, UK and USA. The objectives of the project was originally partly to investigate the growth of waves under fetch-limited condition, and partly to investigate wave transformation from sea to shallow water area. Simultaneous measurements of waves and winds were taken at stations along a line extending 160 km in a westerly direction from the island of Sylt in the Germany Bright.

During the processing of a large number of spectra corresponding to steady easterly wind, the so-called JONSWAP spectrum was obtained

$$S_\eta(f) \;=\; \frac{\alpha g^2}{(2\pi)^4}\, f^{-5}\, exp\left(-\frac{5}{4}\left(\frac{f}{f_m}\right)^{-4}\right)\, \gamma^{exp\left(-\frac{1}{2\sigma^2}\left(\frac{f}{f_m}-1\right)^2\right)} \tag{15}$$

where

$$\begin{aligned}
\alpha &= 0.076\, x^{-0.22} \\
x &= g\, F\, U_{10}^{-2} \\
f_m &= \frac{3.5\, g\, x^{-0.33}}{U_{10}} \\
\sigma &= 0.07 \qquad\qquad f \le f_p \\
\sigma &= 0.09 \qquad\qquad f > f_p
\end{aligned}$$

γ : Peak enhancement coefficient

U_{10} : Wind speed, 10 m above mean water level

The parameterized JONSWAP spectrum reads

$$S_\eta(f) \;=\; \alpha\, H_s^2\, f_p^4\, f^{-5}\, \gamma^\beta\, exp\left(-\frac{5}{4}\left(\frac{f_p}{f}\right)^4\right) \tag{16}$$

$$\alpha \;\approx\; \frac{0.0624}{0.230 \;+\; 0.0336\,\gamma \;-\; \left(\frac{0.185}{1.9\,+\,\gamma}\right)}$$

$$\beta \;=\; exp\left(-\frac{(f\;-\;f_p)^2}{2\,\sigma^2\, f_p^2}\right)$$

$$\sigma \;\approx\; 0.07 \quad f \le f_p$$

$$\sigma \;\approx\; 0.09 \quad f \ge f_p$$

γ: Peak enhancement coefficient

The JONSWAP spectrum is characterized by a parameter γ, the so-called peak enhancement parameter, which controls the sharpness of the spectral peak, cf. Fig.4. In the North Sea the γ value ranges from 1 to 7 with the mean value 3.3.

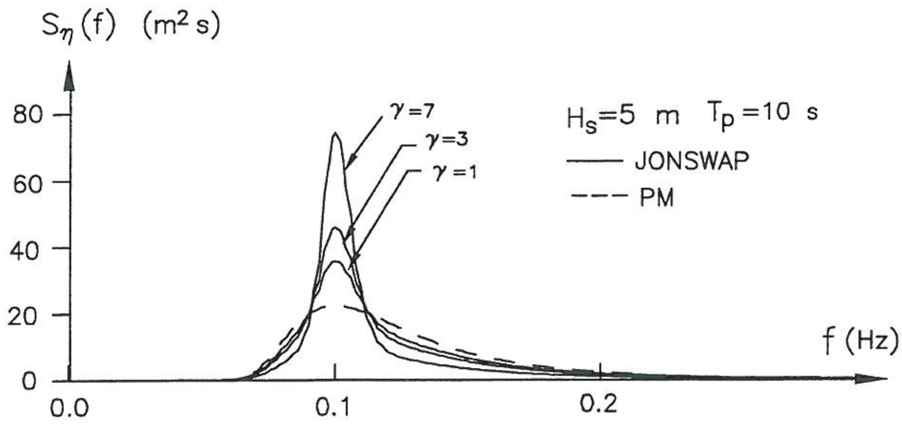


Fig.4. Example of JONSWAP spectrum.

Remarks on standard spectra

Actually wave spectra usually exhibit some deviations from these standard spectra. Concerning the spectrum of swell, the available information is insufficient because many swell records are contaminated by local wind waves. Ochi et al. (1976) presented a spectrum which has two peaks, one associated with swell and the other with locally generated waves. One of the few reports on pure swell spectra indicates that it can be approximately described by the JONSWAP spectrum with relatively larger γ value (Goda 1985). There are still other standard spectra, e.g. Bretsneider (1959), Darbyshire (1952), Scott (1965), Mitsuyasu (1971, 1972) and the ISSC spectrum.

Furthermore, the above mentioned spectra are one-dimensional and valid only for deep water. With respect to shallow water wave spectrum and directional spectrum, many researches are ongoing.

3.4 References

Burcharth, H.F. and Brorsen, M. , 1978. *On the design of gravity structures using wave spectra.* Lecture on Offshore Engineering, Edited by W.J.Graff and P. Thoft-Christensen, Institute of Building Technology and Structural Engineering, Aalborg University, Denmark.

Goda, Y. , 1985. *Random seas and design of marine structures.* University of Tokyo Press, Japan, 1985

Sarpkaya, T. and Isaacson, M. , 1981. *Mechanics of wave forces on offshore structures.* Van Nostrand Reinhold Company, New York, ISBN 0-442-25402-2, 1981.

SPM , 1984. *Shore Protection Manual.* Coastal Engineering Research Center, Waterway Experiment Station, US Army Corps of Engineers, 1984.

3.5 Exercise

1) The wind speed measured at the elevation of 5 m is $U_5 = 20\ m/s$. Calculate $U_{19.5}$ to be used in the PM-spectrum.

2) Convert the fastest mile wind speed $U_f = 29\ m/s$ to twenty-five-minute average wind speed $U_{t=25\ min}$.

3) Calculate H_{m_0} and T_p with

   deep water situation, fetch 200 kilometers, wind speed at $z = 5\ m$ over water surface is 20 m/s over 2 hours.

4 Extreme wave height analysis

The design wave height is often represented by significant wave height. Significant wave height is a random variable. It varies with respect to time and location. If a structure is to be built in a location of sea where a long-term wave height measurement/hindcast is available, the question an engineer must answer is: How to determine the design wave height ?

Extreme wave height analysis gives the answer to that question, i.e. it is a method to determine the design wave height, based on the importance of the structure (design level) and the statistical analysis of a long-term wave height measurements/hindcast.

4.1 Design level: Return period and encounter probability

The design level is represented by return period or encounter probability.

Return period T

To define return period the following notations are used

X	Significant wave height, which is a random variable due to the statistical vagrancy of nature.
x	Realization of X.
$F(x)$	Cumulative distribution function of X, $F(x) = \text{Prob}(X \leq x)$.
t	Number of years of observation of X.
n	Number of observations in a period of t.
λ	Sample intensity, $\lambda = n/t$.

Fig.1 illustrates the cumulative distribution function of X. The non-exceedence probability of x is $F(x)$, or the exceedence probability of x is $(1 - F(x))$. In other words with $(1 - F(x))$ probability an observed significant wave height will be larger than x.

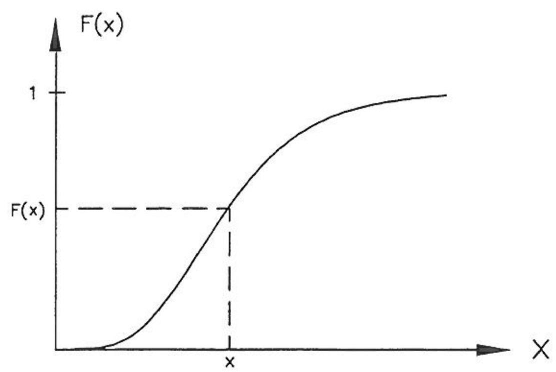


Fig.1. Cumulative distribution function of X.

If the total number of observations is n, The number of observations where $(X > x)$ is

$$k = n\,(\,1 - F(x)\,) = t\,\lambda\,(\,1 - F(x)\,) \tag{1}$$

The return period T of x is defined as

$$T = t\Big|_{k=1} = \frac{1}{\lambda\,(\,1 - F(x)\,)} \tag{2}$$

i.e. on average x will be exceeded once in every T years. x is also called T-year event.

<u>Encounter probability p</u>

Based on the fact that on average x will be exceeded once in every T years, the exceedence probability of x in 1 year is $1/T$. Therefore

non-exceedence probability of x in 1 year	$\text{Prob}(X \leq x)$	$= 1 - \frac{1}{T}$
non-exceedence probability of x in 2 years	$\text{Prob}(X \leq x)$	$= \left(1 - \frac{1}{T}\right)^2$
non-exceedence probability of x in L years	$\text{Prob}(X \leq x)$	$= \left(1 - \frac{1}{T}\right)^L$

and the encounter probability, i.e. the exceedence probability of x within a structure lifetime of L years is

$$p = 1 - \left(1 - \frac{1}{T}\right)^L \tag{3}$$

which in the case of larger T can be approximated

$$p = 1 - exp\left(-\frac{L}{T}\right) \tag{4}$$

<u>Design level</u>

Traditionally the design level for design wave height was the wave height corresponding to a certain return period T. For example, if the design wave height corresponding to a return period of 100 years is 10 m, the physical meaning is that on average this 10 m design wave height will be exceeded once in every 100 years.

In the reliability based design of coastal structures it is better to use encounter probability, i.e. the exceedence probability of the design wave height within the structure lifetime. For example If the structure lifetime L is 25 years, the encounter probability of the design wave height (10 meter) is

$$p = 1 - \left(1 - \frac{1}{T}\right)^L = 22\%$$

This means that this 10 m design wave height will be exceeded with 22% probability within a structure lifetime of 25 years.

4.2 General procedure

In practice engineers are often given a long-term significant wave height measurement/hindcast and required to determine the design wave height corresponding to a certain return period. The general procedure to perform the task is:

1) Choice of the extreme data set based on a long-term wave height measurement/hindcast
2) Choice of several theoretical distributions as the candidates for the extreme wave height distribution
3) Fitting of the extreme wave heights to the candidates by a fitting method. If the least square fitting method is employed, a plotting position formula must be used
4) Choice of the distribution based on the comparison of the fitting goodness among the candidates
5) Calculation of the design wave height corresponding to a certain return period
6) Determination of the confidence interval of the design wave height in order to account for sample variability, measurement/hindcast error and other uncertainties

If structure lifetime and encounter probability are given in stead of return period, we can calculate the return period by eq (3) and proceed as above.

If the followings will be discussed the procedures one by one.

4.3 Data sets

The original wave data are typically obtained either from direct measurements or from the hindcasts based on the meteorological information. Most of the measurements/hindcasts cover a rather short span of time, say less than 10 years in the case of direct measurements and less than 40 years in the case of hindcasts.

In practice three kinds of extreme data sets have been used.

Complete data set	containing all the direct measurements of wave height usually equally spaced in time.
Annual series data set	consisting of the largest wave height in each year of measurements/hindcasts, cf. Fig.2.
Partial series data sets	composed of the largest wave height in each individual storm exceeding a certain level (threshold). The threshold is determined based on the structure location and engineering experience, cf. Fig.2. It is also called POT data set (Peak Over Threshold).

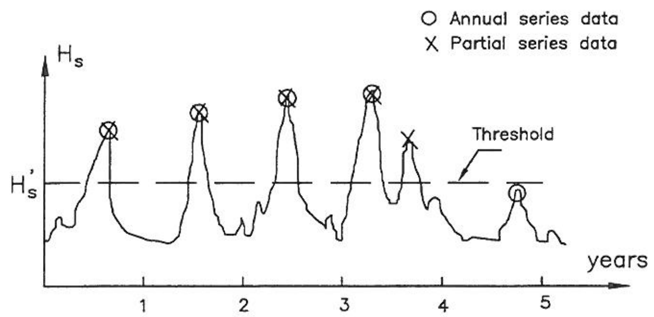


Fig.2. Illustration of the establishment of annual series data set and partial series data set.

The extreme data sets, established based on the original wave data, should fulfill the following 3 conditions:

Independence There must be no correlation between extreme data. The annual series data set and the partial series data set meet the independence requirement because the extreme data are from different storms.

Homogeneity The extreme data must belong to the same statistical population, e.g. all extreme data are from wind-generated waves.

Stationary There must be stationary long-term climatology. Studies of wave data for the North Sea from the last 20 years give evidence of non-stationarity as they indicate a trend in the means. Average variations exist from decades to decades or even longer period of time. However, until more progress is available in investigating long-term climatological variations, the assumption of stationary statistics might be considered realistic for engineering purpose, because the long-term climatological variation is generally very weak.

The complete data set cannot fulfill the required independence between data. Goda (1979) found correlation coefficients of $0.3 - 0.5$ for significant wave heights (measurement duration is 20 minutes and time interval between two succeeding measurements is 24 hours). Moreover, what is interesting in the case of design waves is the wave height corresponding to a very high non-exceedence probability, i.e. the very upper tail of the distribution. If the chosen distribution is not the true one, the very upper tail value will be distorted severely because in the fitting process the chosen distribution will be adjusted to the vast population of the data. For these reasons the complete data set is seldom used.

Most engineers prefer the partial series data set over the annual series data set simply because the former usually gives larger design wave height and hence, more conservatively designed structures.

4.4 Candidate distributions

Generally the exponential distribution, the Weibull distribution, the Gumbel (FT-I) distribution, the Frechet distribution and the Log-normal distribution are the theoretical distributions which fit the extreme wave data well.

$$\text{Exponential} \quad F = F_X(x) = P(X < x) = 1 - e^{-\left(\frac{x-B}{A}\right)} \tag{5}$$

$$\text{Weibull} \quad F = F_X(x) = P(X < x) = 1 - e^{-\left(\frac{x-B}{A}\right)^k} \tag{6}$$

$$\text{Gumbel} \quad F = F_X(x) = P(X < x) = e^{-e^{-\left(\frac{x-B}{A}\right)}} \tag{7}$$

$$\text{Frechet} \quad F = F_X(x) = P(X < x) = e^{-\left(\frac{x}{A}\right)^k} \tag{8}$$

$$\text{Log-normal} \quad F = F_X(x) = P(X < x) = \Phi\left(\frac{\ln(x) - B}{A}\right) \tag{9}$$

where

X	A characteristic wave height, which could be the significant wave height H_s or the one-tenth wave height $H_{\frac{1}{10}}$ or the maximum wave height H_{max}, depending on the extreme data set.
x	Realization of X.
F	Non-exceedence probability of x (cumulative frequency).
$A,\ B,\ k$	Distribution parameters to9 be fitted. In the log-normal distribution A and B are the standard deviation and the mean of X respectively.
Φ	Standard normal distribution function.

No theoretical justification is available as to which distribution is to be used. The author have tried to fit 7 sets of partial series data to all these distributions. These data sets are real data representing deep and shallow water sea states from Bilbao in Spain, Sines in Portugal, the North Sea, Tripoli in Libya, Pozzallo and Follonica in Italy and Western Harbour in Hong Kong. The results show that the Weibull and the Gumbel distributions provide the closest fits. Therefore the following discussion is exemplified with these two distributions.

4.5 Fitting methods and procedure

Four generally applied methods of fitting the extreme data set to the chosen distributions are the maximum likelihood method, the method of moment, the least square method and the visual graphical method. The most commonly used methods are the maximum likelihood method and the least square method.

Least square method

Eqs (6) and (7) can be rewritten as

$$X = A\,Y + B \tag{10}$$

where Y is the reduced variate defined according to the distribution function

$$Y = (-ln(1-F))^{\frac{1}{k}} \quad \text{Weibull distribution} \tag{11}$$

$$Y = -ln(-lnF) \quad \text{Gumbel distribution} \tag{12}$$

The fitting procedure is summarized as the follows:

1) Rearrange the measured/hindcast extreme data (total number n) in the descending order, (x_i), $i = 1, 2, \cdots, n$ (X_1=max).
2) Assign a non-exceedence probability F_i to each x_i by an appropriate plotting position formula (cf. next section), thus obtaining a set of data pairs, $(F_i,\ x_i)$, $i = 1, 2, \cdots, n$.
3) Calculate the corresponding y value by eq (11) or eq (12), thus obtaining a new set of data pairs, $(y_i,\ x_i)$, $i = 1, 2, \cdots, n$.
4) Determine the regression coefficients of eq (10) by

$$A = \frac{Cov(Y,X)}{Var(Y)} \qquad B = \overline{X} - A\overline{Y}$$

$$Var(Y) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \overline{Y}\right)^2$$

$$Cov(Y,X) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \overline{Y})(x_i - \overline{X})$$

$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} y_i \qquad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

In the case of the Weibull distribution various k values are predefined and A and B are fitted accordingly. The final values of the three parameters are chosen based on the fitting goodness.

Maximum likelihood method

The 2-parameter Weibull distribution is

$$\text{Weibull} \quad F(x) = 1 - e^{-\left(\frac{x-x'}{A}\right)^k} \tag{13}$$

where x' is the threshold wave height, which should be smaller than the minimum wave height in the extreme data set. For unexperienced engineers several threshold values can be tried, and the one which produces best fit is finally chosen.

the maximum likelihood estimate k is obtained by solving the following equation by an iterative procedure:

$$N + k \sum_{i=1}^{N} \ln(x_i - x') = N k \sum_{i=1}^{N} \left((x_i - x')^k \ln(x_i - x')\right) \left(\sum_{i=1}^{N} (x_i - x')^k\right)^{-1} \tag{14}$$

The maximum likelihood estimate of A is

$$A = \left[\frac{1}{N} \sum_{i=1}^{N} (x_i - x')^k\right]^{1/k} \tag{15}$$

For the Gumbel distribution, the maximum likelihood estimate of A is obtained by solving the following equation by an iterative procedure:

$$\sum_{i=1}^{N} x_i \exp\left(-\frac{x_i}{A}\right) = \left[\frac{1}{N} \sum_{i=1}^{N} x_i - A\right] \sum_{i=1}^{N} \exp\left(-\frac{x_i}{A}\right) \tag{16}$$

The maximum likelihood estimate of B is

$$B = A \ln\left[N \left(\sum_{i=1}^{N} \exp\left(-\frac{x_i}{A}\right)\right)^{-1}\right] \tag{17}$$

4.6 Plotting position formulae

When the least square method is applied a plotting position formula must be chosen. The plotting position formula is used to assign a non-exceedence probability to each extreme wave height. The plotting position is of special importance when dealing with very small samples.

The non-exceedence probability (F_i) to be assigned to (x_i), can be determined based on three different statistical principles, namely sample frequency, distribution of frequency and order statistics, cf Burcharth et al (1994).

Mean, median and mode

The definition of mean, median and mode of a random variable X is given in the following because they are involved in some of the plotting position formulae.

Take the Gumbel distribution as an example. The distribution function $F_X(x)$ and density function $f_X(x)$ of a Gumbel random variable X reads

$$F_X(x) = P(X < x) = e^{-e^{-\left(\frac{x-B}{A}\right)}} \qquad f_X(x) = \frac{dF_X(x)}{dx} \tag{18}$$

The definition and value of the mean, the median and the mode are

$$\text{Mean} \quad x_{mean} = E[X] = \int_{-\infty}^{+\infty} x f_X(x)dx \approx B + 0.577A \tag{19}$$

$$\text{Median} \quad x_{median} = x\Big|_{F_X(x)=0.5} = B + 0.367A \tag{20}$$

$$\text{Mode} \quad x_{mode} = x\Big|_{f_X(x)=max} = B \tag{21}$$

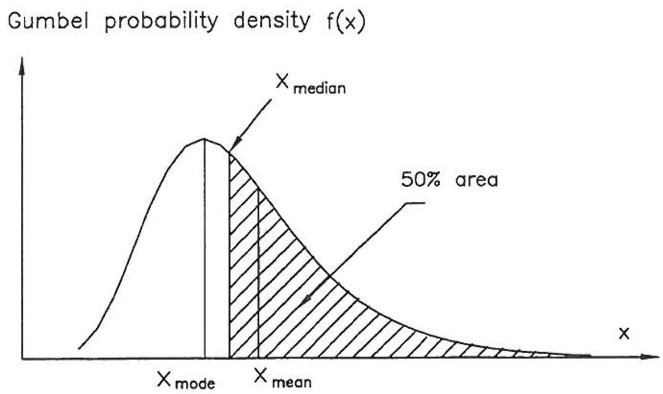


Fig. 3. Mean, median and mode of the Gumbel random variable.

Order statistics

Assume that a random variable X has a cumulative distribution function F_X, and probability density function f_X, i.e.

$$F_X(x) = P(X < x) \tag{22}$$

Furthermore, assume n data sampled from X and arranged in the descending order, x_1 being the largest value in n data.

Here x_1 is one realization of the ordered random variable X_1, defined as the largest value in each sample. The distribution function of X_1 is

$$F_{X_1}(x) = P(X_1 < x) = (F_X(x))^n = (P(X < x))^n \tag{23}$$

$F_{X_1}(x)$ may also be interpreted as the probability of the non-occurrence of the event ($X > x$) in any of n independent trials.

The density function of X, f_X, and the density function of X_1, f_{X_1}, are sketched in Fig.4.

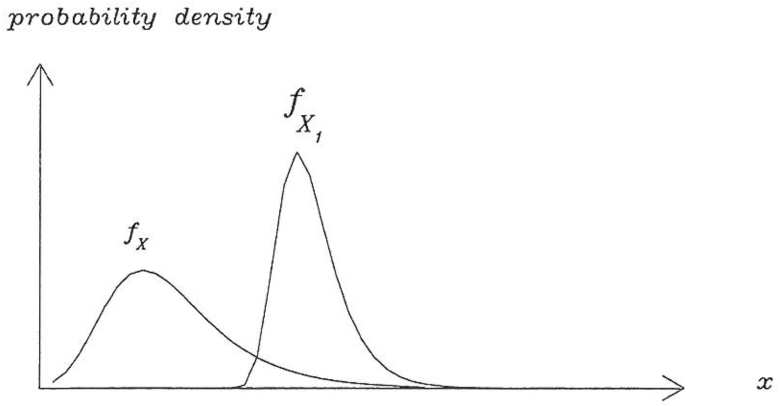


Fig. 4. f_X *: Density function of* X. f_{X_1} *: density function of* X_1.

For other ordered random variables X_i, $i = 2, 3, \cdots, n$, The distribution functions $F_{X_i}(x)$ can also be expressed as the function of $F_X(x)$, cf. Thoft-Christensen et al.(1982).

Plotting position based on sample frequency

This method is based solely on the cumulative frequency of the samples. The widely used formula is the so-called California plotting position formula

$$F_i = 1 - \frac{i}{n} \qquad i = 1,\ 2,\ \cdots,\ n \tag{24}$$

where	x_i	Extreme data in the descending order (x_1=max)
	F_i	Non-exceedence probability of x_i.
	n	Sample size, i.e. total data number.

The disadvantage of this plotting position formula is that the smallest extreme data x_n cannot be used because $F_n = 0$.

Plotting position based on distribution of frequencies

Assume that the random variable X has a cumulative distribution function F_X. The i'*th* highest value in n samples, X_i, is a random variable, too. Consequently, $F_{X_i}(x_i)$, the cumulative frequency of x_i, is a random variable, too. The philosophy of this method is to determine the plotting position of x_i via either the mean, the median or the mode of the random variable $F_{X_i}(x_i)$. The plotting position formula by this method is independent of the parent distribution (distribution-free).

Weibull (1939) used the mean of $F_{X_i}(x_i)$ to determine the cumulative frequency F_i to be assigned to x_i

$$\text{Weibull} \quad F_i == 1 - \frac{i}{n+1} \tag{25}$$

There is no explicit formula for the median of $F_{X_i}(x_i)$. However, Benard (1943) developed a good approximation

$$\text{Benard} \quad F_i \approx 1 - \frac{i - 0.3}{n + 0.4} \tag{26}$$

The plotting position formula based on the mode of $F_{X_i}(x_i)$ has not drawn much attention, because the chance of the occurrence of mode is still infinitesimal even though mode is more likely to occur than the mean and median, Plotting position based on order statistics

The philosophy of this method is to determine the plotting position of x_i via the mean, the median and the mode of the ordered random variable X_i.

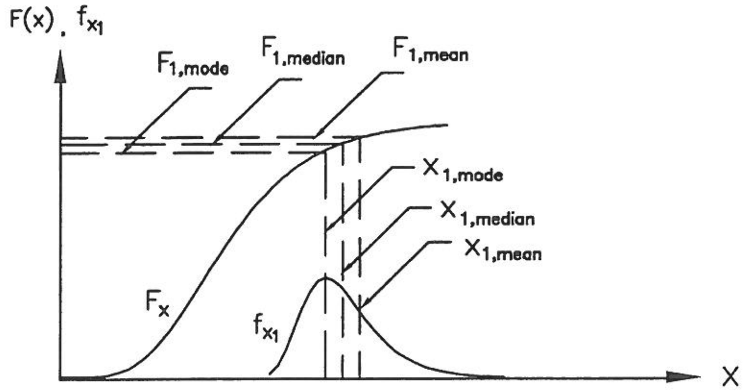


Fig. 5. Illustration of the determination of F_1 based on the mean, the median and the mode of X_1.

Plotting positions based on the mean value are distribution-dependent and not explicitly available. The best known approximations are

Blom	$F_i = 1 - \frac{i-3/8}{n+1/4}$	Normal distribution	(27)
Gringorten	$F_i = 1 - \frac{i-0.44}{n+0.12}$	Gumbel distribution	(28)
Petrauskas	$F_i = 1 - \frac{i-0.3-0.18/k}{n+0.21+0.32/k}$	Weibull distribution	(29)
Goda	$F_i = 1 - \frac{i-0.2-0.27/\sqrt{k}}{n+0.20+0.23/\sqrt{k}}$	Weibull distribution	(30)

The plotting position based on the median value of the ordered random variable is the same as that based on the median value of distribution of frequency.

Summary on plotting position formulae

The choice of the plotting position formula depends on engineer's personnel taste.

From the statistical point of view the plotting position formula based on the mean (unbiased) is preferred because the expected squared error is minimized. Rosbjerg (1988) advocates the choice of the median plotting position formula (Benard formula) because it is distribution-free and is based both on the distribution of frequency and the order statistics. In practice the Weibull plotting position formula is most widely used.

4.7 Fitting goodness

Normally several candidate distributions will be fitted and the *best* one is chosen. The linear correlation coefficient, defined as

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\ Var(Y)}} \tag{31}$$

is widely used as the criterion for the comparison of the fitting goodness. However, ρ is defined in the linear plotting domain (y, x), where the reduced variate y is dependent on the distribution function. Therefore, the interpretation of this criterion is less clear.

With the fitted distribution functions, the wave heights corresponding to the non-exceedence probability of the observed wave heights can be calculated, cf. eqs (33) and (34). The *average relative error E*, defined as

$$E = \frac{1}{n}\sum_{i=1}^{n} \frac{\mid x_{i,estimated} - x_{i,observed} \mid}{x_{i,observed}} \tag{32}$$

is a good simple criterion with a clear interpretation. $E = 5\ \%$ means that on the average, the central estimation of wave height deviates from the observed wave height by 5 %. Obviously a smaller E-value indicates a better fitted distribution.

The statistical hypothesis test can also be used in the comparison of the fitting goodness (Goda et al. 1990)

4.8 Design wave height: x^T

The design wave height x^T is the wave height corresponding to the return period T. The Weibull and Gumbel distributions, eqs (6) and (7), are rewritten as

$$x = A(-ln(1 - F))^{\frac{1}{k}} + B \quad \text{Weibull distribution} \tag{33}$$

$$x = A(-ln(-ln(F))) + B \quad \text{Gumbel distribution} \tag{34}$$

Define the sample intensity λ as

$$\lambda = \frac{\text{number of extreme data}}{\text{number of years of observation}} \tag{35}$$

and employ the definition of return period T

$$T = \frac{1}{\lambda\,(\,1\;-\;F\,)} \quad \text{or} \quad F\;=\;1\;-\;\frac{1}{\lambda T} \tag{36}$$

Inserting eq (36) into eqs (33) and (34), we get (now x means the wave height corresponding to return period T, and therefore is replaced by x^T)

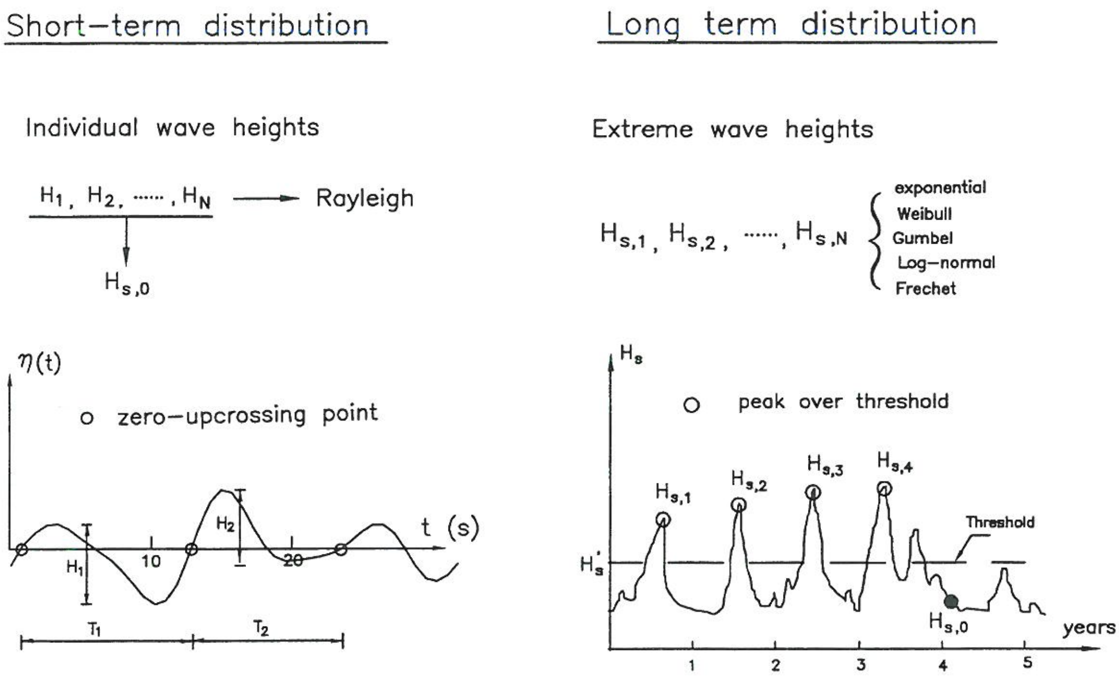
$$x^T \;=\; A\left(-ln(\tfrac{1}{\lambda T})\right)^{\frac{1}{k}} + B \qquad \text{Weibull distribution} \tag{37}$$

$$x^T \;=\; A\left(-ln(-ln(1-\tfrac{1}{\lambda T}))\right) + B \quad \text{Gumbel distribution} \tag{38}$$

where A, B and k are the fitted distribution parameters.

Remark

Some students have a confusion between the short-term distribution of individual wave heights and the long-term distribution of extreme wave heights. The confusion can be cleared by the following figure.



4.9 Example

Delft Hydraulics Laboratory performed a hindcast study for the Tripoli deep water wave climate and identified the 17 most severe storms in a period of 20 years. The ranked significant wave heights are listed in Table 2.

Table 2. Tripoli storm analysis

rank i	Significant wave height x_i (m)	non-exceedence probability F_i	Reduced variate y_i Gumbel	Reduced variate y_i Weibull ($k = 2.35$)
1	9.32	0.944	2.86	1.57
2	8.11	0.889	2.14	1.40
3	7.19	0.833	1.70	1.28
4	7.06	0.778	1.38	1.18
5	6.37	0.722	1.12	1.11
6	6.15	0.667	0.90	1.04
7	6.03	0.611	0.71	0.98
8	5.72	0.556	0.53	0.92
9	4.92	0.500	0.37	0.86
10	4.90	0.444	0.21	0.80
11	4.78	0.389	0.06	0.74
12	4.67	0.333	-0.09	0.68
13	4.64	0.278	-0.25	0.62
14	4.19	0.222	-0.41	0.55
15	3.06	0.167	-0.58	0.49
16	2.73	0.111	-0.79	0.40
17	2.33	0.056	-1.06	0.30

You are required to find the design wave height which has 22% exceedence probability within a structure lifetime of 25 years.

The steps in the analysis are as follows:

1) Calculate the sample intensity by eq (35) $\lambda = \frac{17}{20}$
2) Calculate the return period by eq (3) $T = 100$ years
3) Assign a non-exceedence probability F_i to each observed wave height x_i according to the Weibull plotting position formula. Results are shown in Table 2.
4) Choose the Weibull and the Gumbel distributions as the candidate distributions. Calculate the values of the reduced variate $\{y_i\}$ according to eqs (11) and (12) respectively. For the Weibull distribution $\{y_i\}$ involves the iterative calculation. $\{y_i\}$ of the two distributions are also shown in Table 2.

5) Fit data (y_i, x_i) to eq (10) by the least square method and obtain the distribution parameters:
Weibull, k = 2.35, A = 5.17, B = 0.89
Gumbel, A = 1.73, B = 4.53
The fitting of the data to the Gumbel and the Weibull distributions is shown in Fig. 6.

6) Compare the goodness of fitting according to the value of the average relative error E, eq (32)
E = 4.72 % for the Weibull distribution fitting
E = 6.06 % for the Gumbel distribution fitting
Because of a clearly smaller E-value the Weibull distribution is taken as the representative of the extreme wave height distribution

7) Calculate the wave height corresponding to a return period of 100 years X^{100} by eq (37) $x^{100} = 10.64$ m

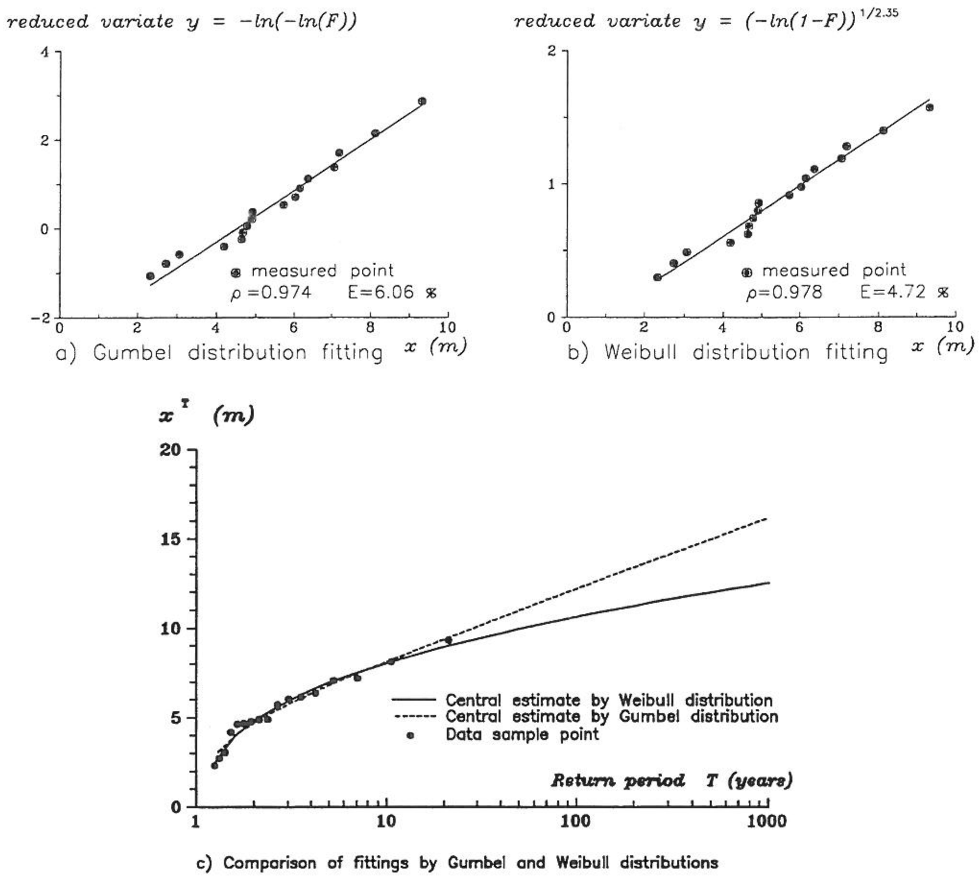


Fig. 6. Fitting to the Gumbel and the Weibull distributions and comparison.

4.10 Sources of uncertainties and confidence interval

Sources of uncertainties

The sources of uncertainty contributing to the uncertainty of the design wave height are:

1) Sample variability due to limited sample size.
2) Error related to measurement, visual observation or hindcast.
3) Choice of distribution as a representative of the unknown true long-term distribution
4) Variability of algorithms (choice of threshold, fitting method etc.
5) Climatological changes

The uncertainty sources 1) and 2) can be considered by numerical simulation in the determination of the design wave height.

Wave data set contains measurement/hindcast error. Measurement error is from malfunction and non-linearity of instruments, such as accelerometer and pressure cell, while hindcast error occurs when the sea-level atmospheric pressure fields are converted to wind data and further to wave data. The accuracy of such conversion depends on the quality of the pressure data and on the technique which is used to synthesize the data into the continues wave field. Burcharth (1986) gives an overview on the variational coefficient C (standard deviation over mean value) of measurement/hindcast error.

Visual observation data should not be used for determination of design wave height because ships avoid poor weather on purpose. With the advance of measuring equipment and numerical model, generally C value has been reduced to below 0.1.

Table 1. variational coefficient of extreme data C

Methods of determination	Accelerometer Pressure cell Vertical radar	Horizontal radar	Hindcast by SPM	Hindcast other	Visual
Variational Coe. C	0.05-0.1	0.15	0.12-0.2	0.1-0.2	0.2

Confidence interval of design wave height x^T

We use an example to demonstrate how the confidence interval of the design wave height is determined. The gumbel distribution curve in Fig.7 is obtained by fitting Tripoli significant wave height to Gumbel distribution by the least square fitting method and the Weibull plotting position formula.

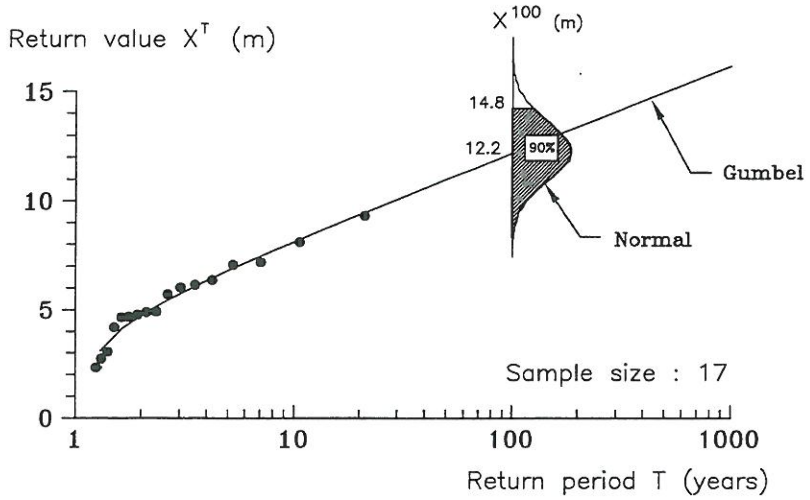


Fig.7. Design wave height.

If the design level for design wave height is a return period of 100 years, i.e. $T = 100$, the design wave height is $x^{100} = 12.2\ m$.

If other uncertainties, e.g. sample variability, is included, the design wave height x^{100} becomes a random variable. The distribution of the design wave height x^{100}, which is usually assumed to follow the normal distribution, can be obtained by numerical simulation to be described in the next section, cf. Fig.7. In order to account sample variability, a confidence band is often applied. For example, the design wave height is 14.8 m which corresponds to the 90% one-sided confidence interval, cf. Fig.7.

Numerical simulation

To exemplify the discussion, it is assumed that the extreme wave height follows the Gumbel distribution

$$F = F_X(x) = P\,(X < x) = exp\left(-exp\left(-(\frac{x-B}{A})\right)\right) \qquad (39)$$

where X is the extreme wave height which is a random variable, x a realization of X, A and B the distribution parameters.

Due to the sample variability and measurement/hindcast error, the distribution parameter A and B become random variables,

In order to account the sample variability and measurement/hindcast error, a numerical simulation is performed as explained in the followings.

A sample with size N is fitted to the Gumbel distribution. The obtained distribution parameters A_{true} and B_{true} are assumed to be the true values.

1) Generate randomly a data between 0 and 1. Let the non-exceedence probability F_1 equal to that data. the single extreme data x is obtained by (cf. Fig.8)

$$x = F_X^{-1}(F_1) = A_{\text{true}}[-\ln(-\ln F_1)] + B_{\text{true}} \tag{40}$$

2) Repeat step 1) N times. Thus we obtain a sample belonging to the distribution of eq (39) and the sample size is N.

3) Fit the sample to the Gumbel distribution and get the new estimated distribution parameters A and B.

4) Calculate the wave height x^T corresponding to the return period T by eq (38)

5) Repeat steps 2) to 4), say, 10,000 times. Thus we get 10,000 values of x^T.

6) Choose the wave height corresponding to the specified confidence band.

In order to include the measurement/hindcast error the following step can be added after step 1). This step is to modify each extreme data x generated by step 1), based on the assumption that the hindcast error follows the normal distribution, cf. Fig.8

1*) Generate randomly a data between 0 and 1. Let the non-exceedence probability F_2 equal to that data. the modified extreme data x_{modified} is obtained by

$$x_{\text{modified}} = x + C\,x\,\Phi^{-1}(F_2) \tag{41}$$

where Φ is the standard normal distribution and C is the coefficient of variation of the measurement/hindcast error. C ranges usually from 0.05 to 0.1.

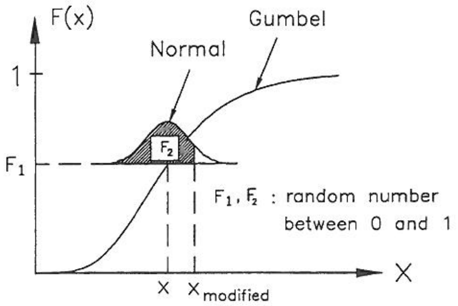


Fig.8. Simulated wave height taking into account measurement/hindcast error.

Example

Again the Tripoli deep water wave data is used as an example to demonstrate the determination of the design wave height and the influence of sample variability.

By fitting the extreme data to Gumbel distribution we obtain the distribution parameters $A = 1.73$ and $B = 4.53$, cf. Fig.9. The design wave height corresponding to a return period of 100 years is 12.2 m.

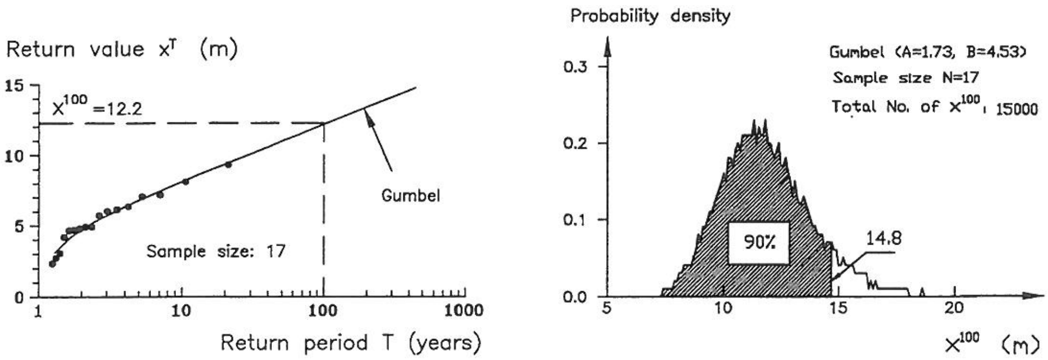


Fig.9. Simulated distribution of x^{100} (sample variability).

If sample variability is included, the design wave height x^{100} becomes a random variable. The distribution of the design wave height x^{100} can be obtained by numerical simulation, cf. Fig.9. In order to account sample variability, an 80% confidence band is often applied. In the case of wave height estimate, one-sided confidence interval is preferred over two-sided confidence interval because the lower bound of the confidence band is of less interest. Therefore, the design wave height is 14.8 m which corresponds to 90% one-sided confidence interval.

4.11 Physical consideration of design wave height

Wave breaking

The design wave height must be checked against wave breaking condition. Wave breaking occurs due to wave steepness (Stokes wave theory) or limited water depth (Solitary wave theory). Based on laboratory and field observations, many empirical formulae for wave breaking condition have been proposed, e.g. Goda (1985).

Structural response characteristics

The choice of design wave height depends not only on the structure life time, but also on the character of the structural response.

Fig.10 indicates as an example the differences in armour layer damage development for various types of rubble structures. The figure illustrates the importance of evaluation of prediction and confidence limits related to the estimated design wave height, especially in case of structures with brittle failure characteristics. To such cases a lower damage level must be chosen for the mean value design sea state. The figure is illustrative. In reality also the confidence bands for the damage curves should be considered.

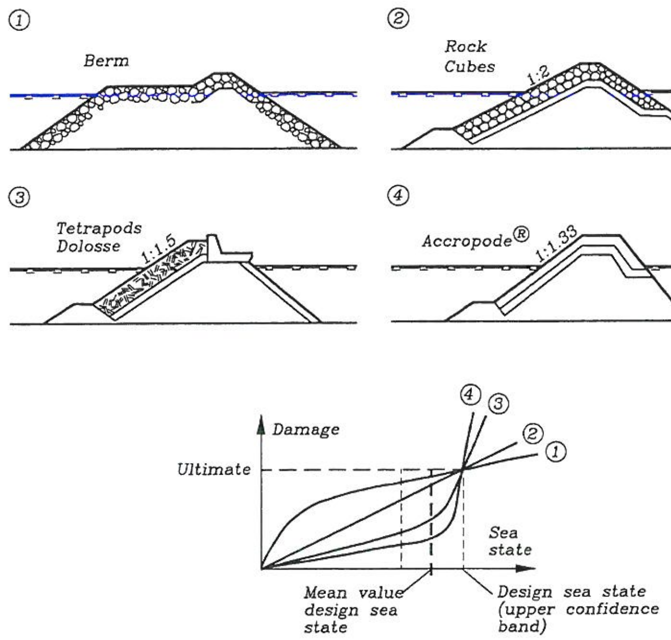


Fig.10. Illustration of typical armour layer failure characteristics for various types of rubble mound structures (Burcharth 1993).

4.12 Wave period

There is no theory to determine the design wave period corresponding to the design wave height obtained by the extreme analysis, due to the complexity and locality of the joint distribution between wave height and wave period.

Fig.11 shows examples of scatter diagrams representing the joint distribution of significant wave height, H_s, and mean wave period, T_m, and still water level, z, respectively. The numbers in the scatter diagrams are the number of observations falling in the corresponding predefined intervals of H_s, T_m and z.

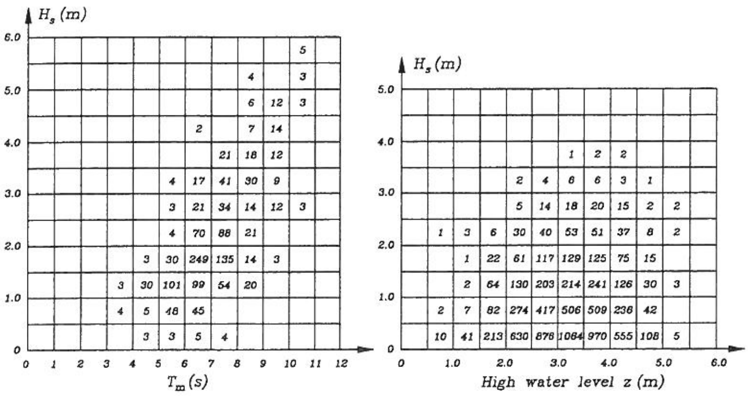


Fig.11. Scatter diagrams signifying examples of joint distributions of H_s and T_m, and H_s and water level, z.

In practice, several wave periods within a realistic range are simply assigned to the design wave height to form the candidates of the design sea state conditions. Then by theoretical consideration and/or laboratory investigation, the one which is most dangerous is chosen.

DS449 gives the range of peak wave period

$$\sqrt{\frac{130\ H_s}{g}} \ < \ T_p \ < \ \sqrt{\frac{280\ H_s}{g}} \tag{42}$$

4.13 Water level

The sea water level is affected by the following effects:

1) Astronomical effect: Tides generated by the astronomical aspect is the best understood due to their extreme regularity and the simplicity of observations. At a site without any previous tidal records usually one or a few month of recording will be sufficient to analyze the astronomical effect on the water level. The astronomical tidal variations can be found in the Admiralty Tide Tables.
2) Meteorological effect: In shallow water the water level is also affected by the meteorological effects, namely,
   i) Barometric: The higher barometric pressure causes a lower water level and vise versa.
   ii) Wind: Strong wind creates a set-up of the water level on the downwind side and a set-down on the upwind side.

   It is difficult to determine the meteorological effect on the water level. If water level records are available for a long period of time, the meteorological effect can be isolated from the astronomical effect and subjected to the extreme analysis in order to establish the long-term statistics of the water level. If such records are not available, numerical models can, using wind and/or barometric chart, give reliable results.
3) Earthquake

The water depth read from the Chart Datum is the one corresponding to the Lowest Astronomical Tide, which is the lowest tide level under the average meteorological conditions, cf. Fig.12, which gives also the widely used terminology and abbreviation of the various sea water levels.

The extreme analysis should be performed on both the high water level and the low water level. Based on the established long-term statistics is given the design low water level and the design high water level.

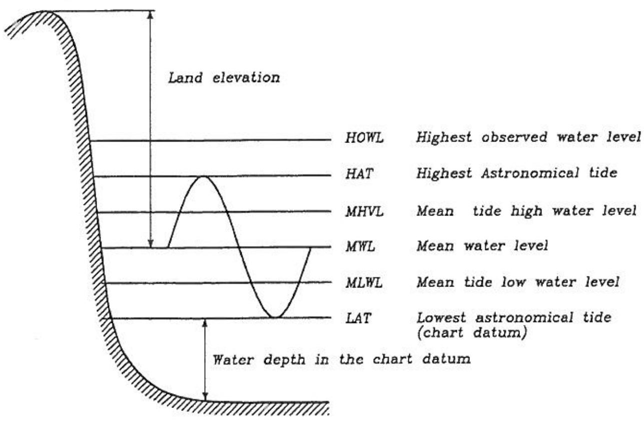


Fig.12. Water depth.

4.14 Multiparameter extreme analysis

A sea state should be characterized at least by some characteristic values of wave height (e.g. H_s), wave period (e.g. T_m), the wave direction, and the water level, because these four parameters are the most important for the impacts on the structures. Of importance is also the duration of the sea state and sometimes also the shape (type) of the wave spectrum.

When more sea state parameters have significant influence on the impact on the structure considerations must be given to the probability of occurrence of the various possible combinations of the parameter values.

Burcharth (1993) proposed the following principle for multiparameter extreme analysis:

For the general case where several variables are of importance but the correlation coefficients are not known the best joint probability approach would be to establish a long-term statistics for the response in question, e.g. for the run-up, the armour unit stability, the wave force on a parapet wall, etc.

If we assume that the variables of importance are H_s, T_m, α (wave direction) and z (water level) then by hindcasting or/and measurements several data sets covering some years can be established

$$(H_{s,i},\ T_{m,i},\ \alpha_i,\ z_i)\,,\quad i = 1,\ 2,\ \ldots n$$

For each data set the response in question is either calculated from formulae or determined by model tests. If for example run-up, R_u, is in question a single variable data set is obtained

$$(R_{u,i})\,,\quad i = 1,\ 2,\ \ldots n$$

The related long-term statistics can be established by fitting to a theoretical extreme distribution (extreme analysis).

4.15 References

Benard, L.R. , 1943. *Statistical analysis in hydrology.* Trans. Am. Soc. Civ. Eng., 108, pp 1110-1160

Burcharth, H.F. , 1986. *On the uncertainties related to the estimation of extreme environmental conditions.* Proceeding of Seminar on Uncertainties Related to the Design and Construction of Offshore Jacket Structures, Copenhagen, 1986, Published by Danish Society of Hydraulic Engineering

Burcharth, H.F. , 1993. *The design of breakwaters.* Department of Civil Engineering, Aalborg University, Denmark, 1993

Burcharth, H.F. and Zhou Liu , 1994. *On the extreme wave height analysis.* Proceedings of HYDRO-PORT'94, Yokosuka, Japan, 19-21 October, 1994

Cunnane, C. , 1978. *Unbiased plotting positions - a review.* J. Hydrology, 37, pp 205-22.

Goda, Y. , 1979. *A review on statistical interpretation of wave data.* Port and Harbour Research Institute, 18(1), 1979

Goda, Y. , 1985. *Random seas and design of marine structures .* University of Tokyo Press, Japan, 1985

Goda, Y. , 1988. *On the methodology of selecting design wave height.* Proc. 21st Int. Conf. on Coastal Engr., Spain.

Goda, Y., Kobune, K. , 1990. *Distribution function fitting for storm wave data.* Proc. 22nd Int. Conf. on Coastal Engr., The Netherlands.

Le Mehaute, B. and Shen Wang , 1984. *Effects of measurement error on long-term wave statistics.* Proceedings of the 19th International Conference on Coastal Engineering, Houston, USA, 1984.

Liu, Z. and Burcharth, H.F , 1996. *Design wave height related to structure lifetime.* Proceedings of the 25th International Conference on Coastal Engineering, Orlando, USA, 1996.

Rosbjerg, D. , 1985. *Estimation in partial duration series with independent and dependent peak values.* Journ. of Hydrology, 76, pp 183-195.

Rosbjerg, D. , 1988. *A defence of the median plotting position.* Progress Report 66, ISVA, Technical University of Denmark

Ross, S.M., , 1987. *Introduction to probability and statistics for engineers and scientists.* John Wiley & Sons, Inc., 1987, PP 245-305. ISBN 0-471-60815-7.

Thoft-Christensen, P. and Michael J. Baker , 1982. *Structural reliability theory and its application.* ISBN 3-540-11731-8, Springer-Verlag, 1982.

Weibull, W. , 1939. *A statistical theory of strength of material.* Ing. Vet. Ak. Handl. (Stockholm), 151.

4.16 Exercise

1) The design wave height for Sines breakwater in Portugal is the significant wave height corresponding to 100 years return period.

   The hindcast study of Sines breakwater wave climate gave the following 17 severest storms in the period of 1970-1985:

   H_s in meter

12.0	10.8	10.7	10.2	10.1	9.8	9.6
9.3	9.3	9.0	8.8	8.1	7.8	7.7
7.3	6.9	6.3				

   Fit the data to Gumbel distribution by the least square method and the maximum likelihood method and calculate the design wave height.

2) Taking into consideration sample variability, Use the Monte-Carlo simulation to draw the probability density function of the design wave height and calculate the upper bound of the design wave height corresponding to 90% confidence.

5 Wave generation in laboratory

The importance of wave generation in laboratory is due to the fact that we cannot describe wave phenomenon (formation, transformation and especially breaking) and wave-structure interactions (wave force, run-up, overtopping etc.) purely by mathematics, and hence model tests play an important role in the design of coastal and offshore structures.

5.1 Principle of wave generator

Fig.1 illustrates the basic concept of wave generators. The input signal is the time series of voltage to be sent to servo mechanism. At the same time the servo mechanism receives information on the position of wave paddle through the displacement censor (feed-back). After the comparison of the input signal with the paddle position, the servo mechanism sends a control signal to the valve of hydraulic pump, which converts the output of the hydraulic pump into the movement of the wave paddle.

In stead of hydraulic pump, electric servo-motor, direct-current motor and hydraulic pulse pump have also been applied. The wave paddle shown in Fig.1 is called piston-type wave paddle. Another popular one is hinged-type wave paddle. In reality there are over 20 types of wave paddles.

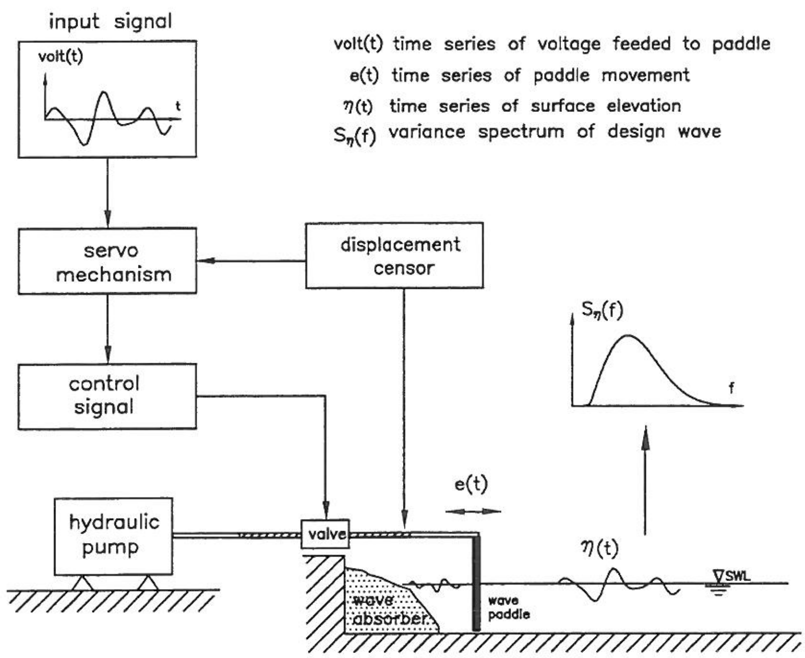


Fig.1. Outline of the wave generator at AaU, Denmark.

5.2 Biésel transfer functions

Biésel transfer functions express the relation between wave amplitude and wave paddle displacement (Biésel et al. 1951).

<u>Formulation</u>

Under the assumption of irrotational and incompressible fluid, the velocity potential produced by paddle movement is formulated in Fig.2.

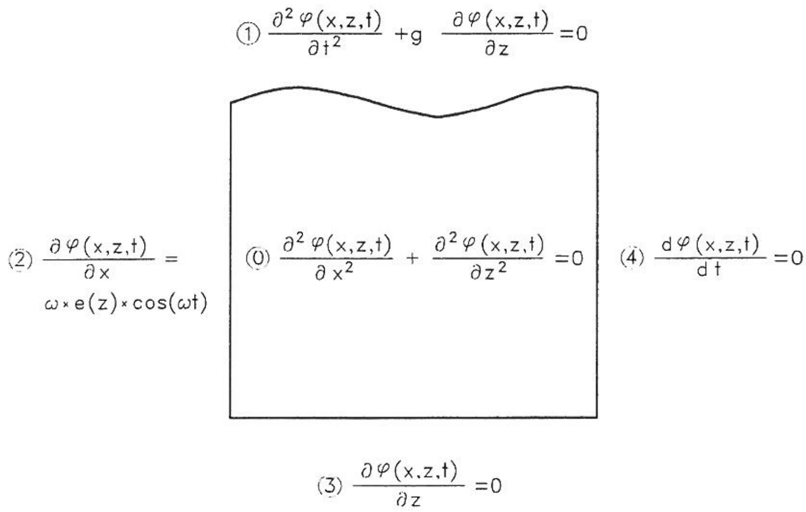


Fig.2. Formulation of boundary value problem.

In Fig.2 the equations express:

0. Laplace equation. Basic equation for potential flow.
1. All water particles at the free surface remain at the free surface (kinematic boundary condition). Free surface is at constant pressure (dynamic boundary condition).
2. The water accompanies the wave paddle. The horizontal velocity of water particle is the same as the paddle. The time series of paddle movement is

$$e(z,t) \;=\; x(z,t)\Big|_{\text{paddle}} \;=\; \frac{S(z)}{2}\,\sin(\omega t) \tag{1}$$

   where $S(z)$ is the stroke of the paddle, cf. Figs 4 and 5.
3. The bottom is impermeable.
4. The propagating wave is of constant form.

Near-field and far-field solution

By solving the boundary value problem the surface elevation in the generated wave field is

$$\eta(x,t) \;=\; (c \cdot \sinh(kh))\,\cos(\omega t - kx) + \left(\sum_{n=1}^{\infty} c_n \sin(k_n h)\; e^{-k_n \cdot x}\right)\,\sin(\omega t) \quad (2)$$

where c, c_n and k_n are coefficients depending on the paddle type, paddle cycling frequency and water depth.

The first term in eq (2) expresses the surface elevation at infinity, by Biésel called the far-field solution, while the second term is the near-field solution. In general only the far-field solution is interesting because the amplitude of a linear wave should not change with location. Fortunately, the "disturbance" from the near-field solution will in a distance of 1-2 water depth from the wave paddle be less than 1% of the far-field solution, cf. Fig.3.

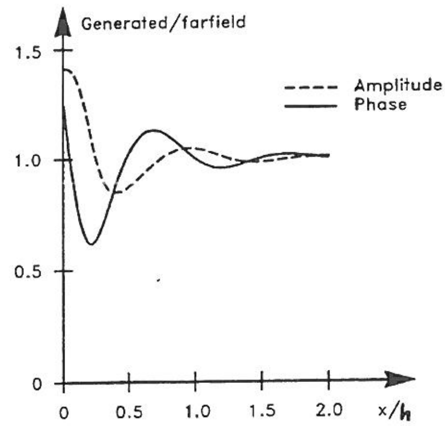


Fig.3. Wave amplitude and phase of the generated wave field relative to the far-field solution. Water depth = 0.7 m and wave period = 0.7 sec

Fig.3 shows that the far-field surface elevation is phase-shifted $\frac{\pi}{2}$ relative to the displacement of the wave paddle. However, because the initial phase of the surface elevation will not change wave properties, the paddle movement is often written as in phase with the surface elevation.

The Biésel transfer function, i.e. the amplitude relation between wave and paddle, is obtained by the far-field solution

$$c(S(z), k, h) \cdot \sinh(kh) \;=\; \frac{H}{2} \quad (3)$$

The Biésel transfer functions for the two most popular wave paddles are given in the followings.

Biésel Transfer Function for piston-type paddle

$$S(z) = S_0$$

$$\frac{H}{S_0} = \frac{2\sinh^2(kh)}{\sinh(kh)\cosh(kh) + kh} \tag{4}$$

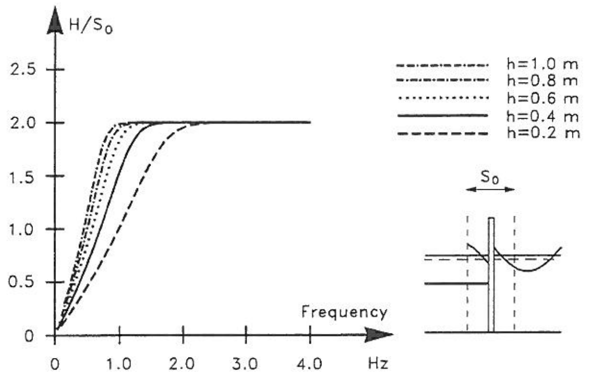


Fig.4. Biésel Transfer Function for piston-type paddle.

Biésel Transfer Function for hinged-type paddle

$$S(z) = \frac{S_0}{h}\cdot(h+z), \qquad S_0 = S(z=0)$$

$$\frac{H}{S_0} = \frac{2\sinh(kh)\,(1-\cosh(kh) + kh\,\sinh(kh))}{kh\,(\sinh(kh)\cosh(kh) + kh)} \tag{5}$$

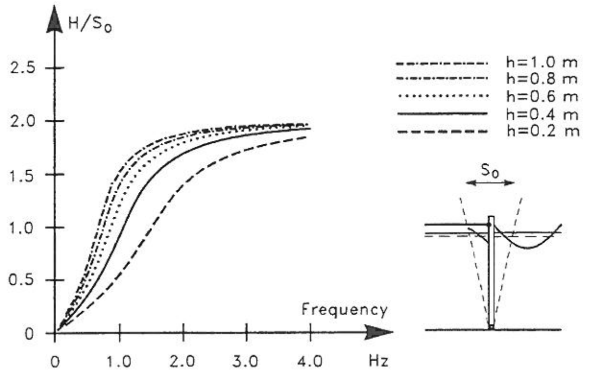


Fig.5. Biésel Transfer Function for hinged-type paddle.

5.3 Examples

Calibration of wave paddle

Before generating waves the wave paddle should be calibrated in order to obtain the calibration coefficient of the paddle. The calibration of wave paddle is performed by sending a signal which increases gradually from 0 to 1 voltage in one minute, and then measuring the wave paddle displacement, cf. Fig.6.

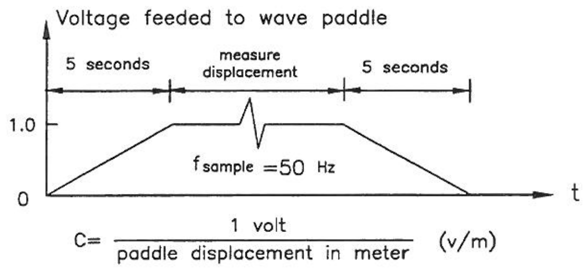


Fig.6. Signal to be sent to wave paddle for calibration.

Modification of signal

In order to avoid a sudden movement of wave paddle, the signal should be modified by a data taper window. Fig. 7 illustrates the principle of the linear data taper window.

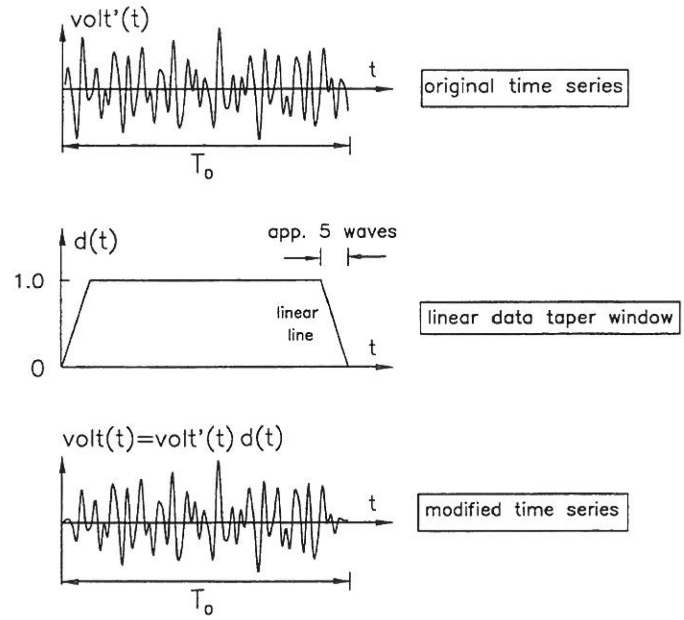


Fig.7. Linear data taper window.

General procedure

The most important aspect of wave generation is the preparation of the input signal corresponding to the variance spectrum of design wave. Fig.8. illustrates one of simple methods.

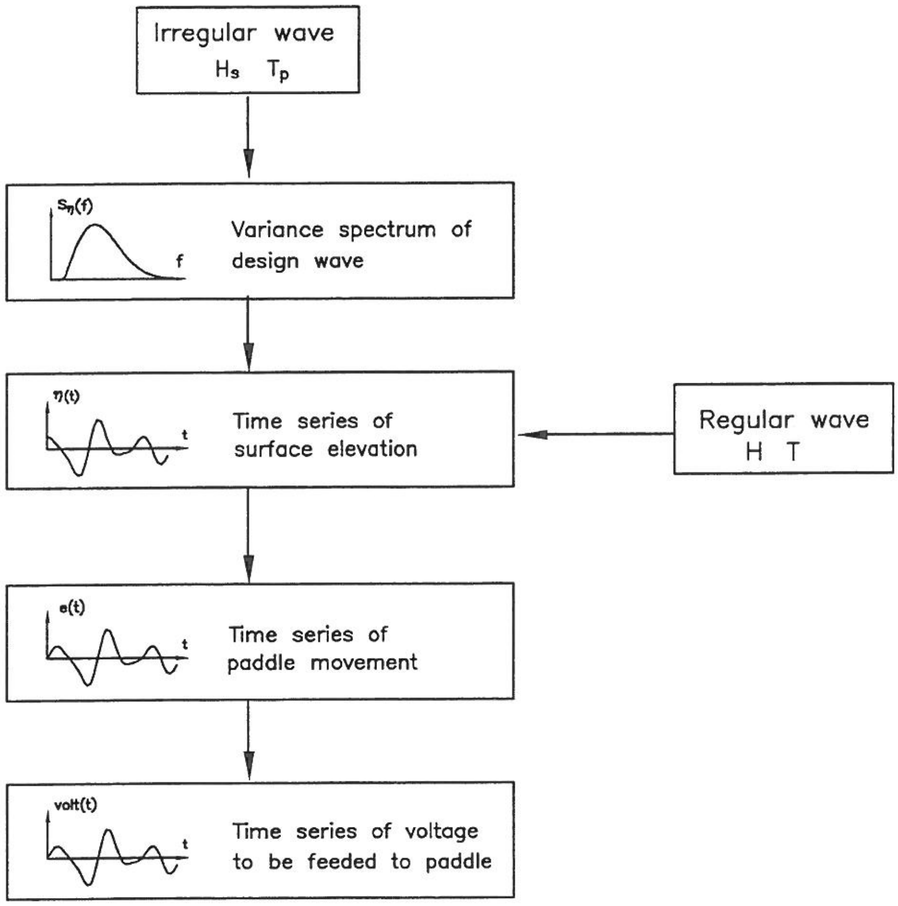


Fig.8. Preparation of input signal.

Of course the recording of the generated wave is necessary so that it can be checked whether the variance spectrum of the generated wave is close to that of design wave, according to Murphy's law[1].

[1] Murphy's law: *Anything, which might go wrong, will go wrong.*

Example : Linear wave generation

Generate a linear wave by piston-type wave paddle.

Wave height	$H = 0.1$	m
Wave period	$T = 1.5$	s
Water depth	$h = 0.4$	m

1) The surface elevation of the linear wave is

$$\eta(t) \;=\; a\ \cos(\omega t + \delta) \;=\; \frac{H}{2}\ \cos(\omega t + \delta)$$

where the angular frequency $\omega \;=\; 2\pi/T \;=\; 4.2\ \mathrm{s}^{-1}$, The initial phase δ is given a random number between 0 and 2π.

2) Convert the time series of surface elevation into the time series of piston movement by the help of Biesel transfer function.

The Biésel transfer function for the piston-type wave paddle

$$B \;=\; \frac{H}{S_0} \;=\; \frac{2\ \sinh^2(kh)}{\sinh(kh)\ \cosh(kh)\ +\ kh} \;=\; 0.80$$

The time series of the piston movement is

$$e(t) \;=\; \frac{S_0}{2}\ \cos(\omega t + \delta) \;=\; \frac{H}{2B}\ \cos(\omega t + \delta)$$

3) Convert the time series of piston movement into the time series of voltage to be feeded to the piston-type wave paddle.

$$volt(t) \;=\; C\ e(t)$$

where C is the calibration coefficient of the wave paddle.

4) Modify the time series of the voltage by the linear data taper window in order to avoid sudden movement of the wave paddle.

5) Sample the data from the modified time series of the voltage at $f_{sample} = 50$ Hz and send the signal to the wave paddle.

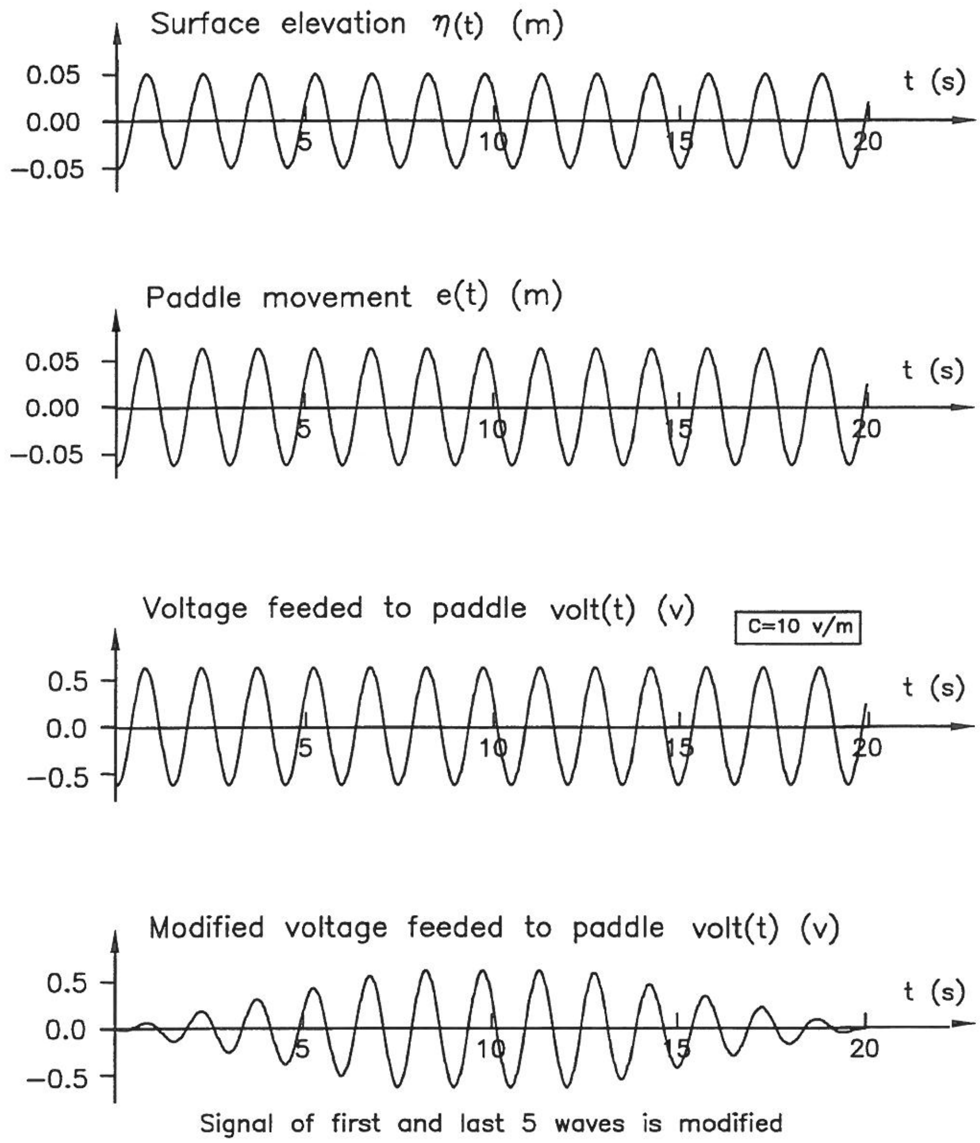


Fig.9. Preparation of input signal for linear wave.

Example : Irregular wave generation

Generate irregular wave by piston-type wave paddle according to JONSWAP spectrum with $H_s = 0.1$ m, $T_p = 1$ s, water depth $h = 0.4$ m and peak enhancement coefficient $\gamma = 3.3$.

JONSWAP spectrum

$$S(f) \;=\; \alpha \, H_s^2 \, f_p^4 \, f^{-5} \, \gamma^{\beta} \, exp\left(-\tfrac{5}{4}\left(\tfrac{f_p}{f}\right)^4\right)$$

$$\alpha \;\approx\; \frac{0.0624}{0.230 + 0.0336\,\gamma - \left(\frac{0.185}{1.9+\gamma}\right)}$$

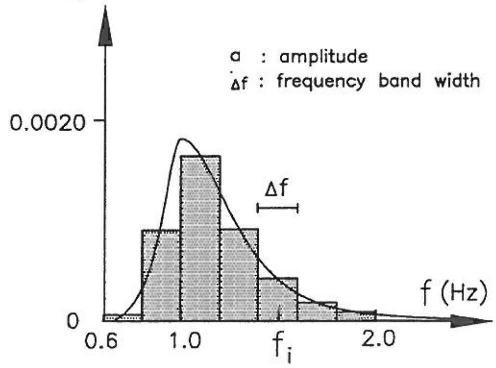
$$\beta \;=\; exp\left(-\frac{(f-f_p)^2}{2\,\sigma^2\,f_p^2}\right)$$

$\sigma \approx 0.07 \quad f \leq f_p$

$\sigma \approx 0.09 \quad f \geq f_p$

γ : peak enhancement coefficient

$$S_\eta(f) = \frac{\frac{1}{2}a^2}{\Delta f} \quad (\mathrm{m^2\,s})$$



1) Draw the JONSWAP spectrum with the specified parameters.
   Note $S_\eta(f_p) = 0.00193$ ($\mathrm{m^2}$ s).

2) Divide the spectrum evenly into N parts in the interval (f_{start}, f_{stop}).
   To ensure accuracy usually

$$N \geq 50 \qquad S_\eta(f_{start}) \leq 0.01\; S_\eta(f_p) \qquad S_\eta(f_{stop}) \leq 0.01\; S_\eta(f_p)$$

For the sake of simplicity in this example

$$N = 7 \qquad f_{start} = 0.6 \text{ Hz} \qquad f_{stop} = 2.0 \text{ Hz}$$

The frequency band width

$$\Delta f \;=\; \frac{f_{stop} - f_{start}}{N} \;=\; 0.2 \text{ Hz}$$

That is to say, the irregular wave is composed of 7 linear waves. The surface elevation of the irregular wave is

$$\eta(t) \;=\; \sum_{i=1}^{7} \eta_i(t) \;=\; \sum_{i=1}^{7} a_i \;\cos(\omega_i t + \delta_i)$$

3) Determine the angular frequency ω_i, amplitude a_i and initial phase δ_i of each linear waves.

The frequency of each linear wave is

$$f_i = f_{start} + i\,\Delta f - \frac{\Delta f}{2} \qquad i = 1, 2, \cdots, 7$$

The angular frequency is

$$\omega_i = \frac{2\pi}{T_i} = 2\pi f_i \qquad i = 1, 2, \cdots, 7$$

The variance of each linear wave is

$$S_\eta(f_i)\,\Delta f = \frac{1}{2} a_i^2 \qquad i = 1, 2, \cdots, 7$$

Therefore the amplitude is

$$a_i = \sqrt{2\,S_\eta(f_i)\,\Delta f} \qquad i = 1, 2, \cdots, 7$$

$S(f_i)$ is calculated from the JONSWAP spectrum. The initial phase δ_i is assigned a random number between 0 and 2π.

4) Convert the time series of surface elevation into the time series of piston movement by the help of Biesel transfer function.

The Biésel transfer function for the piston-type wave paddle

$$B_i = \frac{H_i}{S_{0,i}} = \frac{2\,\sinh^2(k_i h)}{\sinh(k_i h)\,\cosh(k_i h) + k_i h} \qquad i = 1, 2, \cdots, 7$$

The time series of the piston movement is

$$e(t) = \sum_{i=1}^{7} \frac{S_{0,i}}{2} \cos(\omega_i t + \delta_i) = \sum_{i=1}^{7} \frac{H_i}{2B_i} \cos(\omega_i t + \delta_i)$$

i		1	2	3	4	5	6	7
f_i	(Hz)	0.7	0.9	1.1	1.3	1.5	1.7	1.9
T_i	(s)	1.4	1.1	0.9	0.8	0.7	0.6	0.5
L_i	(m)	2.5	1.7	1.2	0.9	0.7	0.5	0.4
ω_i	(/s)	4.4	5.7	6.9	8.2	9.4	10.7	11.9
k_i	(/m)	2.6	3.6	5.0	6.9	9.1	11.6	14.5
$S_\eta(f_i)$	(m^2s)	0.00007	0.00079	0.00103	0.00036	0.00021	0.00012	0.00007
a_i	(m)	0.0052	0.0178	0.0203	0.0119	0.0092	0.0070	0.0055
H_i	(m)	0.0103	0.0356	0.0406	0.0239	0.0183	0.0141	0.0110
δ_i		1.2	0.7	0.3	2.5	6.1	4.3	4.1
B_i		1.00	1.36	1.69	1.90	1.98	2.00	2.00

5) Convert the time series of piston movement into the time series of voltage to be feeded to the piston-type wave paddle.

$$volt(t) \;=\; C\ e(t)$$

where C is the calibration coefficient of the wave paddle.

6) Modify the time series of the voltage by the linear data taper window in order to avoid sudden movement of the wave paddle.

7) Sample the data from the modified time series of the voltage at $f_{sample} = 50$ Hz and send the signal to the wave paddle.

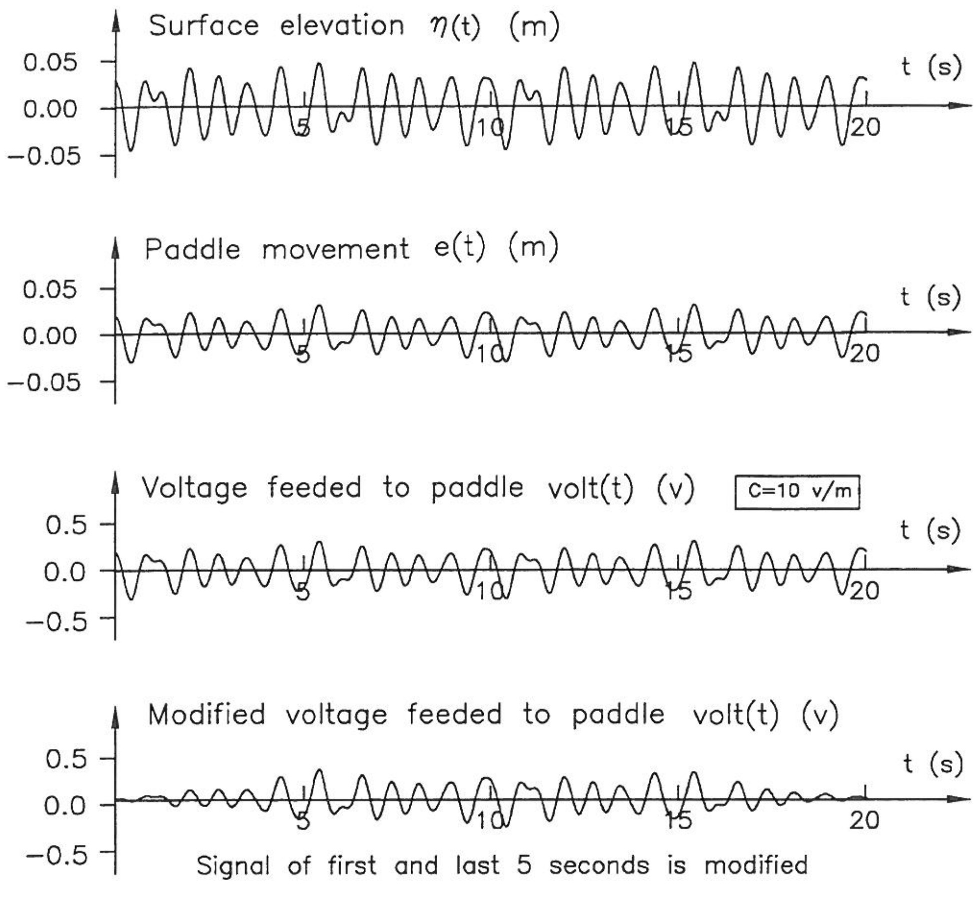


Fig.10. Preparation of input signal for irregular wave.

Remarks

For more complicated wave generation method, including active wave absorption and 3-D wave generation, reference is made to Frigaard et al. (1993). All these aspects have been implemented in a user-friendly software package named PROFWACO (Frigaard et al. 1993). With respect to wave generation techniques we are proud of the fact that the Hydraulic & Coastal Engineering Laboratory of Aalborg University is one of the leading institutes in the world.

5.4 References

Dean, R.G. and Dalrymple, R.A. , 1991. *Water wave mechanics for Engineers and scientists.* Second printing with correction, World Scientific Publishing Co. Pte. Ltd., Singapore, 1991.

Frigaard, P., Høgedal, M. and Christensen, M. , 1993. *Wave generation theory.* Hydraulic & Coastal Engineering Laboratory, Department of Civil Engineering, Aalborg University, Denmark, 1993.

Frigaard, P., Høgedal, M. and Christensen, M. , 1993. *PROFWACO, Users Guide.* Hydraulic & Coastal Engineering Laboratory, Department of Civil Engineering, Aalborg University, Denmark, 1996.

Goda, Y. , 1985. *Random seas and design of marine structures.* University of Tokyo Press, Japan, 1985

5.5 Exercise

1) Estimate the minimum power of motor needed for generating a linear wave with Wave height $H = 0.20$ m, wave period $T = 1.5$ s, water depth $h = 0.4$ m, wave flume width $B = 1.5$ m.

2) What is the minimum distance between the wave gauge and wave paddle ?

3) Explain the importance that the wave paddle can produce sufficiently large wave height. List the factors which limit the maximum wave height obtainable in a wave flume.

4) Make a computer program for examples given in lecture.